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ON MESO-SCALE MOUNTAIN WAVES ON THE ROTATING EARTH

By

ARNT ELIASSEN

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Summary. A stationary, meso-scale, small-amplitude disturbance of a straight baroclinic air current on the rotating earth is studied on the basis of quasi-static linearized equations. It is shown that an undamped train of stationary, horizontal inertia waves will exist on the lee side, representing a drain of wave energy downstream. The wavelength is proportional to the current velocity and will thus change with height, producing a vertical shear which increases indefinitely downstream. It is suggested that a transition to turbulence will take place at some considerable distance from the mountain.

Introduction. Most of the extensive literature about mountain waves in the atmosphere concerns waves of the internal gravity type, where the horizontal length scale (and hence, the time scale) is sufficiently short so that the earth's rotation can be ignored. Also, long mountain waves of the Rossby type have been the subject of several studies. However, neither of these theories is applicable if one wants to determine the characteristics of air flow over a mountain range of horizontal scale of the order 100—1000 km, e.g. for the purpose of calculating the distribution of meso-scale orographic precipitation. In this case, the relevant waves will be of the mixed gravity-inertia type, and they are sufficiently long to be treated as quasi-static.

From the linear theory of such waves, A. ELIASSEN and PALM (1961, Ch. II) established a relationship between the fluxes of wave energy, heat, and momentum in a vertical plane normal to the current. This relationship is valid when the disturbance vanishes far upstream as well as far downstream.

Recently, JONES (1967) studied linearized gravity-inertia waves, and found that rotation is important near the levels where the particle frequency equals twice the angular velocity, since the governing differential equation is singular at these levels.

The present paper deals with the stationary disturbance produced in a baroclinic current flowing across a meso-scale mountain ridge on the rotating earth. Using the linearized quasi-static equations, and performing, as usual, a Fourier transformation of the variables, a singularity of the type discussed by Jones will appear at the level where there is resonance between the particle frequency and the inertial frequency. The singular level will change with the wave number. The main objective of the study is to clarify what effect this singularity will have upon the composite disturbance produced by a mountain ridge.

The objection might be raised that possible short-wave baroclinic instability (GREEN, 1960) is eliminated *a priori* when the disturbance is required to be stationary. If effective, such instability would make a stationary state impossible. However, the author assumes that, for the scales considered, such instability is too weak to be felt within the adjustment time necessary for a transient motion to approach a steady state within a limited area of the order 1000—2000 km.

2. Mathematical formulation. We shall consider a stationary wave disturbance of a baroclinic current and assume the horizontal length scale of the disturbance to be sufficiently large so that the quasi-static approximation applies. The equations may then be written in pressure coordinates. On the other hand, we assume that the scale of the disturbance is small enough so that the curvature of the earth may be ignored and the Coriolis parameter f may be considered constant. These assumptions restrict the wave spectrum to the band from wave length 100 km to 1000 km, approximately.

The basic current. Let x, y denote horizontal cartesian coordinates along and normal to the current; let z denote height, p pressure, f Coriolis parameter, R gas constant, T temperature, $\kappa = c_p/c_v$ specific heat ratio, and p_0 ($=1$ bar) a reference pressure. The basic state is characterized by the velocity distribution $U(p)$, assumed independent of y, and the distribution of specific volume $\alpha(y, p)$. These are related to the geopotential $\Phi(y, p)$ through the geostrophic and hydrostatic relations:

$$fU = -\frac{\partial \Phi}{\partial y} \tag{2.1}$$

$$\alpha = -\frac{\partial \Phi}{\partial p} \tag{2.2}$$

The distribution of potential temperature $\Theta(y, p)$ in the basic state is obtained from

$$\Theta(y,p) = \frac{R}{p_0}\left(\frac{p}{p_0}\right)^{1/\kappa} \alpha(y,p) \tag{2.3}$$

The baroclinicity of the basic state is characterized by

$$\frac{\partial \alpha}{\partial y} = \frac{\alpha}{\Theta}\frac{\partial \Theta}{\partial y} = f\frac{dU}{dp} \tag{2.4}$$

which is a function of p only. As a measure of the static stability of the basic state, we take the quantity

$$\sigma = -\frac{\alpha}{\Theta}\frac{\partial \Theta}{\partial p} = -\left(\frac{\partial}{\partial p} + \frac{1}{\kappa p}\right)\alpha \tag{2.5}$$

Although this quantity will vary with both y and p, its y-dependency is typically quite weak compared with the pronounced variation with p. For the sake of simplicity, we shall here consider σ as a function of p alone.

We may express σ in terms of the buoyancy frequency (Väisälä-Brunt frequency) $N=\left(\frac{g}{\Theta}\frac{\partial \Theta}{\partial z}\right)^{\frac{1}{2}}$ and the pressure scale height $H=-\frac{1}{p}\frac{\partial p}{\partial z}=\frac{RT}{g}$:

$$\sigma = \left(\frac{HN}{p}\right)^2 \tag{2.6}$$

In this equation, the variation of H with height is relatively unimportant, so that H may be considered constant.

The linear perturbation equations. The disturbance will be assumed to be independent of the y coordinate. In the perturbed state, the velocity components in the x-direction and y-direction are $U(p)+u(x,p)$, $v(x,p)$; the individual rate of change of pressure is $\omega(x,p)$; the geopotential is $\Phi(y,p)+\phi(x,p)$, and the specific volume $\alpha(y,p)-\phi_p(x,p)$ (subscripts will be used to denote partial derivatives).

The horizontal equations of motion, the equation expressing conservation of potential temperature, and the continuity equation become after linearization:

$$(Uu+\phi)_x - fv + U_p\omega = 0 \tag{2.7}$$

$$(Uv)_x + fu = 0 \tag{2.8}$$

$$(U\phi_p)_x - fU_p v + \sigma\omega = 0 \tag{2.9}$$

$$u_x + \omega_p = 0 \tag{2.10}$$

In this system, U and σ are functions of p only, while f is constant.

Eliminating u, v, and ϕ between (2.7—2.10), we find the following fourth order equation in ω:

$$U^2\omega_{ppxx} + f^2\omega_{pp} - 2f^2\frac{U_p}{U}\omega_p + (\sigma - UU_{pp})\omega_{xx} = 0 \tag{2.11}$$

Boundary conditions. The elevation of the ground surface will be assumed to be independent of y. The simplest formulation of the boundary condition at the ground is obtained by ignoring the slope of the isobaric surface compared with the slope of the ground, thus considering the ground as a given surface in pressure coordinates:

$$p = p_G - P(x) \tag{2.12}$$

where p_G denotes a constant mean pressure at sea level. The linearized boundary condition at the ground then is

$$\omega(x, p_G) = -U(p_G)P_x \tag{2.13}$$

To obtain a unique solution, we need in addition a radiation condition at the top of the atmosphere. Since waves generated below will acquire a large amplitude as they propagate upward through the high atmosphere, we cannot apply the linearized equations all the way up to $p=0$. We must therefore leave out the layer above $p=p_T$, say, and apply the radiation condition at the level $p=p_T$.

3. Flux of energy and momentum. The wave energy equation is obtained by multiplying (2.7) by u, (2.8) by v, (2.9) by $\sigma^{-1}\phi_p$, (2.10) by ϕ, and adding:

$$\left[\tfrac{1}{2}\left(u^2+v^2+\frac{1}{\sigma}\phi_p^2\right)U+\phi u\right]_x+(\phi\omega)_p = U_p\frac{f}{\sigma}v\phi_p - U_p u\omega \tag{3.1}$$

Here $\frac{1}{2}(u^2+v^2)$ is the kinetic wave energy and $(1/2\sigma)\phi_p{}^2$ the potential wave energy per unit mass. The left-hand side of (3.1) is the flux divergence, and the right-hand side the rate of production of wave energy. In this context, "production" in reality means transformation of basic current potential, internal and kinetic energy into wave energy.

We shall consider the disturbance generated by a mountain ridge of finite extent in the x-direction, and assume that the disturbance vanishes far upstream, i.e. for $x \to -\infty$ (we assume $U(p)>0$ at all levels).

We integrate the wave energy equation (3.1) with respect to x between $-\infty$ and $+\infty$. With the notation

$$\overline{(\;)} = \int_{-\infty}^{+\infty} (\;)\,dx \tag{3.2}$$

we find

$$\left(\frac{U}{2}(u^2+v^2)+\frac{U}{2\sigma}\phi_p^2+\phi u\right)_{x=+\infty}+(\overline{\phi\omega})_p = U_p\frac{f}{\sigma}\overline{v\phi_p} - U_p\overline{u\omega} \tag{3.3}$$

provided that the integrals exist. The first term on the left is the wave energy which escapes downstream. The second term is the integrated vertical flux divergence; hence,

$$F = \overline{\phi\omega} \tag{3.4}$$

is the integrated vertical wave energy flux, reckoned positive downward.

On the right, we have the integrated production terms. We note that wave energy is produced (i) by horizontal eddy flux of sensible heat towards the cold side of the current, and (ii) by vertical eddy flux of momentum in the direction of decreasing velocity. Both production terms are seen to vanish in a barotropic current.

We shall derive another expression for the vertical wave energy flux. Introducing the particle displacements in the y-direction, $\eta(x, p)$, reckoned from their upstream positions, we have in linear approximation

$$v = U\eta_x \tag{3.5}$$

Likewise, we may write

$$\omega = U\Pi_x \tag{3.6}$$

where $\Pi(x, p)$ is the change in pressure for a particle from its upstream value. When these expressions are substituted in (2.7), (2.9), and (2.10), these equations can be integrated to yield

$$Uu + \phi - fU\eta + UU_p\Pi = 0 \tag{3.7}$$

$$U\phi_p - fUU_p\eta + \sigma U\Pi = 0 \tag{3.8}$$

$$u + (U\Pi)_p = 0 \tag{3.9}$$

where the integration constants vanish since all variables vanish far upstream.

Multiplying (3.7) by ω, (3.8) by $(f/\sigma)v$, and subtracting, we find

$$\begin{aligned} Uu\omega + \phi\omega - \frac{f}{\sigma}Uv\phi_p &= -\frac{f^2}{\sigma}UU_p\eta v + fU(\eta\omega + \Pi v) - UU_p\Pi\omega \\ &= -U\left(\frac{f^2}{2\sigma}UU_p\eta^2 - fU\eta\Pi + \tfrac{1}{2}UU_p\Pi^2\right)_x \end{aligned} \tag{3.10}$$

Integration with respect to x yields

$$F = \overline{\phi\omega} = U\left(\frac{f}{\sigma}\overline{v\phi_p} - \overline{u\omega}\right) - U\left(\frac{f^2}{2\sigma}UU_p\eta^2 - fU\eta\Pi + \tfrac{1}{2}UU_p\Pi^2\right)_{x=\infty} \tag{3.11}$$

If u, v, ϕ, and ϕ_p all vanish, not only far upstream but also far downstream, then the integrated energy equation (3.3) gives

$$F_p = (\overline{\phi\omega})_p = U_p\left(\frac{f}{\sigma}\overline{v\phi_p} - \overline{u\omega}\right) = \frac{\partial}{\partial p}\left[U\left(\frac{f}{\sigma}\overline{v\phi_p} - \overline{u\omega}\right)\right] - U\frac{\partial}{\partial p}\left(\frac{f}{\sigma}\overline{v\phi_p} - \overline{u\omega}\right) \tag{3.12}$$

Moreover, if η and Π vanish far downstream, we obtain from (3.11)

$$F = \overline{\phi\omega} = U\left(\frac{f}{\sigma}\overline{v\phi_p} - \overline{u\omega}\right) \tag{3.13}$$

Under the conditions when (3.12) and (3.13) are both valid, it follows that

$$U\frac{\partial}{\partial p}\left(\frac{f}{\sigma}\overline{v\phi_p}-\overline{u\omega}\right)=0 \tag{3.14}$$

so that

$$\frac{F}{U}=\frac{f}{\sigma}\overline{v\phi_p}-\overline{u\omega}=\text{ constant with height} \tag{3.15}$$

in a layer where U has no zero.

This theorem (or rather the more general theorem for the case $\partial/\partial y \neq 0$) was proved by A. Eliassen and Palm (1960). However, their proof depended upon the condition $U_p \neq 0$, whereas the above derivation shows that this condition is not necessary.

Using (3.8), (3.5), and (3.6), equation (3.15) may also be written

$$\frac{F}{U}=\overline{(f\eta-u)\omega}=\text{ constant with height} \tag{3.16}$$

The quantity on the right may be interpreted as the vertical flux of angular momentum, shown to be constant with height by Jones (1967).

4. Fourier transformation of the variables. As usual in such problems, we introduce the Fourier transform:

$$\omega(x,p)=\text{Re}\int_0^\infty \hat{\omega}(k,p)e^{ikx}dk \tag{4.1}$$

and similar expressions for the other variables.

From (2.11), we find the equation to be satisfied by $\hat{\omega}$:

$$\left(U^2-\frac{f^2}{k^2}\right)\hat{\omega}_{pp}+2\frac{f^2}{k^2}\frac{U_p}{U}\hat{\omega}_p+(\sigma-UU_{pp})\hat{\omega}=0 \tag{4.2}$$

This equation has regular singularities at the level where $U(p)=0$ (the critical level) and at the levels where $U(p)=\pm f/k$. We shall assume here that $U>0$ at all levels, so that there is only one relevant singularity, $p=p_s$, defined by

$$U(p_s)=\frac{f}{k} \tag{4.3}$$

Within the range of wave lengths considered (100—1000 km), f/k is of the order 10 m/s, so that the singularity will actually appear within the layer $p_T<p<p_G$ for normal values of the wind velocity.

From the Fourier transform of (2.7), (2.8), and (2.10), we find the following expressions for $\hat{u}$, $\hat{v}$, and $\hat{\phi}$ in terms of $\hat{\omega}$:

$$\hat{u}=\frac{i}{k}\hat{\omega}_p \tag{4.4}$$

$$\hat{v}=-\frac{f}{k^2U}\hat{\omega}_p \tag{4.5}$$

$$\hat{\phi}=\frac{i}{kU}\left[\left(\frac{f^2}{k^2}-U^2\right)\hat{\omega}_p+UU_p\hat{\omega}\right] \tag{4.6}$$

Wave energy flux. The integrated vertical energy flux $\overline{\phi\omega}$ may be expressed by Parseval's formula:

$$F=\overline{\phi\omega}=\pi\text{Re}\int_0^\infty \hat{\phi}\hat{\omega}^*dk \tag{4.7}$$

where the asterisk denotes the complex conjugate. A sufficient criterion for the validity of this formula is that the integrals $\int_{-\infty}^{+\infty}|\phi|^2dx$ and $\int_{-\infty}^{+\infty}|\omega|^2dx$ both converge. Assuming this to be the case, and using (4.6), it follows that the contribution to F from wave number k is:

$$\hat{F}=\pi\text{Re}(\hat{\phi}\hat{\omega}^*)=\frac{\pi U}{k}\left(1-\frac{f^2}{k^2U^2}\right)\text{Im}[\omega_p\omega^*] \tag{4.8}$$

where Im denotes the imaginary part. If u, v, ϕ, and ϕ_p all tend to zero as $x\rightarrow+\infty$, so that no energy escapes downstream, then (3.15) is also valid; the corresponding equation in the Fourier transforms is

$$\frac{\hat{F}}{U}=\frac{\pi}{k}\left(1-\frac{f^2}{k^2U^2}\right)\text{Im}[\hat{\omega}_p\hat{\omega}^*]=\text{constant with height} \tag{4.9}$$

Solution for a layer with constant U and N. In a layer of constant wind velocity U and constant buoyancy frequency N, we substitute for σ the expression (2.6). Ignoring the slow variation of H with height, the solutions of (4.2) may be written

$$\hat{\omega}=Ap^{\frac{1}{2}+\mu}+Bp^{\frac{1}{2}-\mu} \qquad \text{when } k<\frac{f}{U} \tag{4.10}$$

$$\hat{\omega}=Ap^{\frac{1}{2}+i\lambda}+Bp^{\frac{1}{2}-i\lambda} \qquad \text{when } k>\frac{f}{U} \tag{4.11}$$

where A and B are constants (functions of k), and

$$\mu^2=-\lambda^2=\tfrac{1}{4}+\frac{N^2H^2}{\dfrac{f^2}{k^2}-U^2} \quad : \quad \mu>0\,,\ \lambda>0 \tag{4.12}$$

From (4.9), the corresponding vertical wave energy flux is

$$\hat{F}=\frac{2\pi\mu}{kU}\left(\frac{f^2}{k^2}-U^2\right)\mathrm{Im}[A^*B] \qquad \text{when} \quad k<\frac{f}{U} \tag{4.13}$$

$$\hat{F}=\frac{\pi\lambda}{kU}\left(U^2-\frac{f^2}{k_2^2}\right)(|A|^2-|B|^2) \qquad \text{when} \quad k>\frac{f}{U} \tag{4.14}$$

Thus, when $k>f/U$, $p^{1/2+i\lambda}$ gives a positive (downward), and $p^{1/2-i\lambda}$ a negative (upward) energy flux.

5. Solution near the singularity U = f/k. At the singularity $U=f/k$ in equation (4.2), the particle frequency Uk becomes equal to the inertial frequency f. We shall study the behaviour of the solution in the vicinity of this singularity.

For this purpose, we may develop the coefficients of (4.2) in power series of $p-p_s$, p_s being the singular level defined by (4.3). Instead of $p-p_s$, however, we shall here use the non-dimensional quantity

$$q(p)=1-\frac{f^2}{k^2U(p)^2} \tag{5.1}$$

Assuming $U_p\neq 0$ at the singularity, q will have an ordinary zero there. When q is introduced as new independent variable, i.e.

$$\frac{d}{dp}=q_p\frac{d}{dq}\,,\quad \frac{d^2}{dp^2}=q_p^2\frac{d^2}{dq^2}+q_{pp}\frac{d}{dq} \tag{5.2}$$

equation (4.2) assumes the form

$$q\hat{\omega}_{qq}+\left(1+\frac{qq_{pp}}{q_p^2}\right)\hat{\omega}_q+\frac{1}{q_p^2}\left(\frac{\sigma}{U^2}-\frac{U_{pp}}{U}\right)\hat{\omega}=0 \tag{5.3}$$

If we write the coefficients as power series in q and substitute $\hat{\omega}=q^r\sum_{n=0}^{\infty}a_nq^n$, we find that the indicial equation has the double root $r=0$. A fundamental system of solutions may be expressed as

$$\hat{\omega}_1 = 1 + \sum_{n=1}^{\infty} a_n q^n$$

$$\hat{\omega}_2 = \omega_1 \cdot \ln q + \sum_{n=1}^{\infty} b_n q^n \tag{5.4}$$

The general solution of (5.3) (and hence, of (4.2)), valid where $U(p) > 0$, may be written

$$\hat{\omega} = A[(1+F(q)) \ln q + G(q)] + B(1+F(q)) \tag{5.5}$$

where A and B are complex constants, and where the functions F and G are analytic and satisfy the conditions

$$F(0) = G(0) = 0 \tag{5.6}$$

Thus, near the singular level $q=0$, we may write

$$\hat{\omega} = A \ln q + B + 0\,(q \ln q) \tag{5.7}$$

From (5.1), we have

$$\frac{f}{kU} = (1-q)^{\frac{1}{2}}, \quad q_p = 2(1-q)\frac{U_p}{U} = 2\frac{k}{f}(1-q)^{3/2} U_p = 2\frac{k}{f} U_p(p_s) + 0(q) \tag{5.8}$$

and consequently we get, by differentiation of (5.7),

$$\hat{\omega}_p = q_p \hat{\omega}_q = 2\frac{k}{f} U_p(p_s)\frac{A}{q} + 0(\ln q) \tag{5.9}$$

Substituting (5.7) and (5.9) into (4.4—4.6), we find the corresponding expressions for the other variables:

$$\hat{u} = \frac{2i}{f} U_p(p_s)\frac{A}{q} + 0\,(\ln q) \tag{5.10}$$

$$\hat{v} = -\frac{2}{f} U_p(p_s)\frac{A}{q} + 0\,(\ln q) \tag{5.11}$$

$$\hat{\phi} = \frac{iU_p}{k}[A(\ln q - 2) + B] + 0\,(q \ln q) \tag{5.12}$$

Thus we find that $\hat{\omega}$ and $\hat{\phi}$ have logarithmic singularities at $q=0$, while $\hat{u}$ and $\hat{v}$ have poles.

These formulae hold on either side of the singularity. The question then arises how the solutions on the two sides are to be connected; in other words, we must determine the proper change of $\arg(q)$ as we pass from one side of the singularity to the other.

The analogous problem for the branch-point singularity $U(p)=0$ (the critical level) has been dealt with recently by BOOKER and BRETHERTON (1967). They point out that the solution of the corresponding initial value problem leads to the inversion formula for the Laplace transform, taken along a parallel to the imaginary axis on its positive side, and that, in the Fourier wave number plane, this would correspond to moving the singularity $U=c$ away from the real axis by letting the phase velocity c have a positive imaginary part. This determines the sense of the change in argument from one side of the critical level to the other.

The same method applies in the present case; thus we move the singularity $q=0$ to a point defined by

$$U=\frac{k}{f}+ic_i\,;\quad 0<c_i<<\frac{f}{k} \tag{5.13}$$

$$q_s=\frac{\left(U+\frac{f}{k}\right)\left(U-\frac{f}{k}\right)}{U^2}\approx 2i\frac{k}{f}c_i \tag{5.14}$$

This means that q in (5.7) and (5.9—5.12) must be replaced by $[q-2i(k/f)c_i]$. For $q>>2(k/f)c_i$, we choose $\arg[q-2i(k/f)c_i]=0$. As q decreases along the real axis, $\arg[q-2i(k/f)c_i]$ decreases continuously from 0 to $-\pi$.

The assymptotic stationary solution for large t is obtained by letting $c_i\to 0$ from the positive side. Thus we have

$$\arg(q)=\begin{cases}0 & \text{when } q>0\\ -\pi & \text{when } q<0\end{cases} \tag{5.15}$$

and

$$\ln q=\begin{cases}\ln|q|, & q>0\\ \ln|q|-i\pi, & q<0\end{cases} \tag{5.16}$$

We are now in the position to calculate from (4.8) the contribution from wave number k to the total vertical wave energy flux:

$$\hat{F}=\frac{2\pi}{k}U_p(p_s)q\,\mathrm{Im}\left\{\left[\frac{A}{q}+0(\ln q)\right]\left[A^*(\ln q)^*+B^*+0(q\ln q)\right]\right\}$$

or

$$\hat{F}=\begin{cases}\frac{2\pi}{k}U_p(p_s)\mathrm{Im}(AB^*)+0\,(q(\ln q)^2) & \text{when } q>0\\ \frac{2\pi}{k}U_p(p_s)[\mathrm{Im}(AB^*)+\pi|A|^2]+0\,(q(\ln q)^2) & \text{when } q<0\end{cases} \tag{5.17}$$

Thus the flux is discontinuous at $q=0$. Since $\hat{F}$ is reckoned positive downwards, and since $q<0$ below the singular level when $U_p<0$, and above it when $U_p>0$, it follows that the term $(2\pi^2/k)U_p|A|^2$, which appears for $q<0$, is always directed *towards* the singular level. Therefore the jump in $\hat{F}$ across the singularity has always the sense of a flux convergence, and we may write

$$\Delta\hat{F}=\hat{F}(p_s+0)-\hat{F}(p_s-0)=-\frac{2\pi^2}{k}|U_p|\,|A|^2 \tag{5.18}$$

6. Effect of the singularity upon the solution. In the preceding section, we have considered the Fourier-transformed variables as functions of p (or q) for a given k; in particular, we have considered their variation near the singularity $p=p_s$ ($q=0$). The constants A and B appearing in (5.7), (5.9—5.12), and (5.18) must be determined from the boundary condition at the ground and the radiation condition at the top; they will turn out to be functions of k.

To obtain the solution at a particular level, we must keep p fixed and perform an integration over wave number as shown by (4.1).

We shall not attempt here to give the full solution of the problem, but only study what conclusions can be drawn from knowledge of the behaviour of the Fourier-transformed variables near the singularity, as expressed by (5.7) and (5.9—5.12).

When (5.7) is substituted into (4.1), we obtain

$$\omega=Re\int_0^\infty (A(k)\ln q+B+0\,(q\ln q))e^{ikx}dk \tag{6.1}$$

The integral is taken over all real positive values of k with p kept constant. As k varies, the singular level p_s defined by (4.3) will vary too, and coincide with the chosen fixed p for $k=k_s(p)$, where

$$k_s(p)=\frac{f}{U(p)} \tag{6.2}$$

From the definition of q in (5.1), we have

$$q(k,p)=\frac{k^2-k_s(p)^2}{k^2} \tag{6.3}$$

Thus q has an ordinary zero at $k=k_s$.

We assume that A and B are bounded and absolutely integrable. Then the integral (6.1) can be performed right across the singularity, and converges absolutely. Hence, by the Riemann-Lebesgue lemma,

$$\lim_{x\to\pm\infty}\omega=0 \tag{6.4}$$

By the same argument, it follows from (5.12) that

$$\lim_{x\to\pm\infty} \phi=0 \tag{6.5}$$

The matter is different for u and v, however. Substituting (5.10) into the Fourier integral for u and using (6.3), we find

$$u(x,p)=\mathrm{Re}\frac{2i}{f}\int_0^\infty\left[U_p(p_s(k))\frac{A(k)k^2}{k^2-k_s^2}+0(\ln q)\right]e^{ikx}dk \tag{6.6}$$

where p_s is defined as a function of k by (4.3). The integral is ambiguous because of the pole at $k=k_s$. To overcome this, we invoke Booker & Bretherton's argument referred to above; thus, in accordance with (5.13), the singularity $k=k_s$ is moved to

$$k=\frac{f}{U(p)-ic_i}=\frac{f}{U^2+c_i^2}(U+ic_i)\,,\quad c_i>0 \tag{6.7}$$

This shows that the integral (6.6) should be taken *below* the singular point $k=k_s$.

The integral (6.6) can, of course, not be evaluated without knowing the exact form of the integrand; but the information we have about the singularity suffices to determine the asymptotic form of $u(x, p)$ for large $|x|$.

We consider the expression

$$u_\infty(x,p)=\mathrm{Re}\frac{i}{f}U_p(p)A(k_s)k_s\int_0^\infty\frac{e^{ikx}}{k-k_s}dk \tag{6.8}$$

Since, by (4.3) and (6.1), $p_s(k_s)=p$, it will be seen from (6.6) and (6.8) that $u-u_\infty$ is defined by an integral which converges absolutely. Thus, from the Riemann-Lebesgue lemma,

$$\lim_{|x|\to\infty}(u-u_\infty)=0 \tag{6.9}$$

and $u(x, p)$ may for large $|x|$ be evaluated from (6.8).

The integral (6.8) is obtained by integration around a closed contour of the form shown in Fig. 1a for $x<0$, and in Fig. 1b for $x>0$. For large $|x|$, it follows that

$$u_\infty(x,p)=\begin{cases}0 \text{ when } x<0\\ -2\pi\frac{U_p(p)}{U(p)}\mathrm{Re}[A(k_s)e^{ifx/U(p)}] \text{ when } x>0\end{cases} \tag{6.10}$$

In the same way, we find from (5.11)

$$v_\infty(x,p)=\begin{cases}0 \quad \text{when} \quad x<0 \\ 2\pi\dfrac{U_p}{U}\mathrm{Im}[Ae^{ifx/U}] \quad \text{when} \quad x>0\end{cases} \tag{6.11}$$

From (6.4), (6.10), and (6.11), it follows that all dependent variables tend to zero as $x\to-\infty$, in accordance with the assumption made in section 3. For $x\to+\infty$, ω and ϕ tend to zero (this is necessary for the existence of the integrated vertical wave energy flux $\overline{\phi\omega}$), whereas there is an undamped train of inertia waves in the variables u and v.

It is noteworthy that although ϕ and ω vanish for large positive x, ϕ_p and ω_p do not; from (5.9) it follows

$$\omega_{p,\infty}(x,p)=-\frac{f}{U}v_\infty(x,p) \tag{6.12}$$

Moreover, from (5.10) and (5.12) it will be seen that

$$\phi_{p,\infty}(x,p)=U_p u_\infty(x,p) \tag{6.13}$$

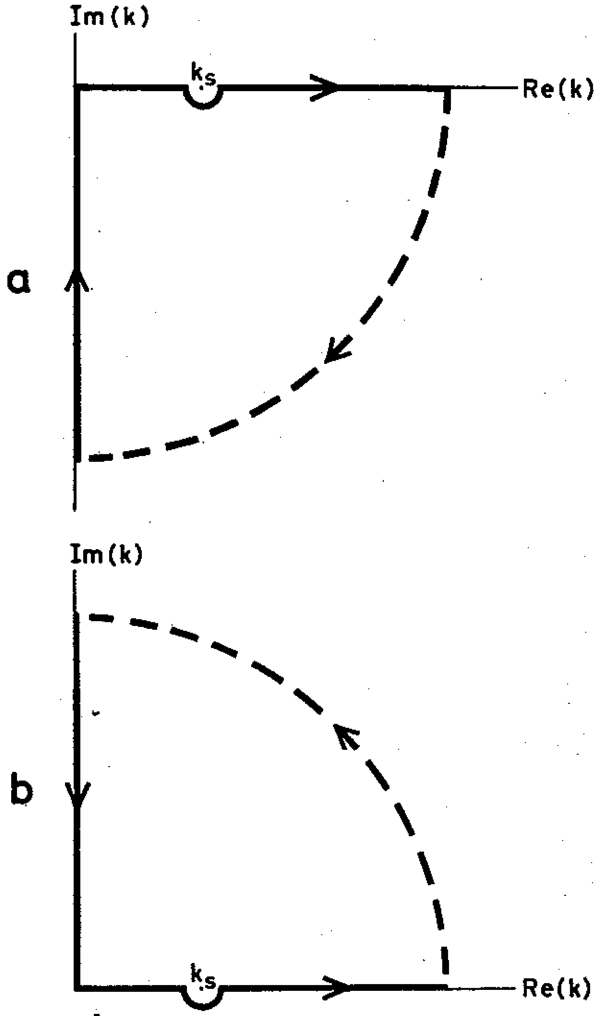

Fig. 1. Paths of integration in the complex k-plane. a: $x<0$. b: $x>0$.

When these expressions are integrated with respect to p, the result is zero by interference, because the phase varies rapidly with height for large positive x; thus, we find again (6.4) and (6.5).

It will be seen that the asymptotic solutions for large positive x given above satisfy the system (2.7—2.10).

Wave energy flux. Clearly, the undamped wave-train for large positive x must represent a drain of wave energy. This drain is represented in equation (3.3) by the terms in the brackets on the left.

From (6.10) and (6.11), we find the following expression for the kinetic wave energy escaping downstream:

$$\frac{U}{2}(u^2+v^2)_{x=+\infty}=2\pi^2\frac{U_p^2}{U}|A|^2 \tag{6.14}$$

It is interesting to compare this expression with the integrated vertical energy flux convergence, which is the last term on the left of (3.3).

From (4.7),

$$F=\overline{\phi\omega}=\int_0^\infty \hat{F}(k,p)dk \tag{6.15}$$

It has been shown that $\hat{F}$ must have the form (5.17). In this formula, A and B are functions of k, p_s is a function of k defined by (4.3), and q a function of k and p defined by (5.1). $\hat{F}$ has a jump for $q=0$; in the integral over k, this jump takes place at $k=k_s(p)$, according to (6.2) and (6.3)· Thus we write

$$F=\overline{\phi\omega}=\int_0^{ks} \hat{F}(k,p)dk+\int_{ks}^\infty \hat{F}(k,p)dk \tag{6.16}$$

Differentiating with respect to p, we find

$$F_p=(\overline{\phi\omega})_p=\frac{dk_s}{dp}[\hat{F}(k_s-0,p)-\hat{F}(k_s+0,p)]+\left(\int_0^{ks}+\int_{ks}^\infty\right)\frac{\partial\hat{F}}{\partial p}dk \tag{6.17}$$

Using (6.2) and (5.18), and remembering that the jump in $\hat{F}$ is always a flux convergence, we obtain

$$F_p=(\overline{\phi\omega})_p=-2\pi^2\frac{U_p^2}{U}|A|^2+\left(\int_0^{ks}+\int_{ks}^\infty\right)\frac{\partial\hat{F}}{\partial p}dk \tag{6.18}$$

We may look upon the first term on the right as that (negative) part of the vertical energy flux divergence which is related to the singularity, whereas the integral to the right is a flux convergence which is present even without the singularity.

Comparison between (6.18) and (6.14) shows that the vertical energy flux convergence due to the singularity equals the horizontal flux of kinetic energy far downstream. Thus, as a result of the occurrence of the singularity, part of the wave energy escapes downstream rather than being radiated to higher levels.

Concerning the remaining terms in the energy equation (3.3), we may make the following brief remarks:

From (6.5), $(u\phi)_{x\to\infty}=0$. The term $(U/2)(\phi_p{}^2)_{x\to\infty}$ is seen to oscillate with wave number $2k_s$, but these oscillations are balanced by similar oscillations in the production term $U_p(f/\sigma)\overline{v\phi_p}$, considered as a function of the upper integration limit. On the other hand, the momentum flux integral $\overline{u\omega}$ occurring in the last term seems to converge (however, the author cannot prove it). Note that the Parseval formula does not apply to the two integrals on the right-hand side of (3.3).

7. Final remarks. We have seen that the earth's rotation will have a marked influence upon meso-scale mountain waves in a baroclinic current, causing a drain of energy downstream in the form of an undamped train of inertia waves. To find the amplitude of these inertia waves, one would have to determine the functions F and G

of equation (5.5) for specified velocity and temperature profiles, and apply the boundary condition at the ground and the radiation condition at the top. No such solutions are given in the present paper.

Certain conclusions may be drawn, however, even without carrying through the calculations in detail.

According to (6.10) and (6.11), the horizontal wave length of the downstream inertia wave is

$$L = 2\pi \frac{U(p)}{f} \tag{7.1}$$

If in (6.10, 6.11) we hold x constant and consider the variation of u and v with height, we find that they oscillate with a "local" vertical wave length

$$M = \left| \frac{d}{dz}\left(\frac{\arg A}{2\pi}\right) - \frac{U_z}{U}\frac{x}{L} \right|^{-1} \tag{7.2}$$

For large x, this expression tends toward zero. Remembering that the amplitude does not change with x, we conclude that the vertical velocity gradients will increase indefinitely with increasing distance downstream.

According to (6.13), a similar oscillation along the vertical is found in the temperature disturbance, which is proportional to ϕ_p. Therefore, vertical temperature gradients of both signs will exist, and their magnitude will increase indefinitely with x.

Clearly, such a motion cannot exist as a stationary, laminar flow; at a certain distance downstream, the vertical velocity gradients will have become large enough so that the motion field is unstable. A transition of the wave motion into turbulence would be a likely result.

It seems likely, however, that quite a long distance downstream from a mountain would be required for such excessive gradients to form. If in equation (7.2) we ignore the contribution from $(d/dz)(\arg A)$, and set $U/U_z \sim 10$ km, $L \sim 600$—1000 km, then x would have to be several thousand kilometers before M is as small as a few kilometers. Still, we cannot quite rule out the possibility that the mechanism described may in certain cases lead to the formation of clear air turbulence.

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