AN EXPLICIT SOLUTION OF THE PROBLEM OF WAVE MOTION IN THREE BAROTROPIC FLUID STRATA

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On Professor Bjerknes' suggestion I have taken up and given the complete solution of the problem of wave motion in three compressible fluid strata, following the method which he had developed in the case of two strata. He gave the solution of this problem in his lectures at the University of Leipzig in the winter term 1914—15, and has kindly placed his manuscript of these lectures at my disposal. After I had thus solved the problem for the case of *three strata*, Professor Bjerknes has now in the preceding paper*) of these publications given the *general formulae for any number of strata* and an explicit solution for the case of two strata. It seems consequently unnecessary to repeat here the author's first complete solution, which has afterwards by Bjerknes been developed in greater generality. We will therefore limit ourselves to give here the explicit solution, purely algebraic, for the case of three strata using the notations of Bjerknes. This case is of a certain interest, while it represents the highest number of strata for which an explicit solution seems practically possible.

The horizontal displacements in the three strata I, II, III are (Bjerknes, p. 16)

$$\xi^{I} = A^{I} f(x-ct)\,, \qquad \xi^{II} = A^{II} f(x-ct)\,, \qquad \xi^{III} = A^{III} f(x-ct)\,, \tag{1}$$

and the corresponding vertical displacements

$$\zeta_1 = B^{I} f'(x-ct)\,, \qquad \zeta_2 = B^{II} f'(x-ct)\,, \qquad \zeta_3 = B^{III} f'(x-ct)\,. \tag{2}$$

The problem admits three velocities of propagation c_1, c_2, c_3, of which the squares c_1^2, c_2^2, c_3^2 are the three roots of the cubic equation (cf. Bjerknes, p. 16)

$$\begin{vmatrix} \gamma_1^{I} - c^2 & \gamma_2^{II} & \gamma_3^{III} \\ \gamma_2^{I} & \gamma_2^{II} - c^2 & \gamma_3^{III} \\ \gamma_3^{I} & \gamma_3^{II} & \gamma_3^{III} - c^2 \end{vmatrix} = 0\,. \tag{3}$$

The quantities γ are

$$\begin{aligned} \gamma_1^{I} &= (a_2^{I} - a_2^{II} + a_3^{II} - a_3^{III} + a_4^{III})(p_2 - p_1)\,. & \gamma_2^{II} &= (a_3^{I} - a_3^{III} + a_4^{III})(p_3 - p_2)\,. \\ \gamma_2^{I} &= (a_3^{II} - a_3^{III} + a_4^{III})(p_2 - p_1)\,. & \gamma_3^{II} &= a_4^{III}(p_3 - p_2)\,. \\ \gamma_3^{I} &= a_4^{III}(p_2 - p_1)\,. & \gamma_3^{III} &= a_4^{III}(p_4 - p_3)\,. \end{aligned} \tag{4}$$

p denoting pressures and a specific volumes as in Professor Bjerknes' paper.

*) V. Bjerknes: On quasi static wavemotion in barotropic fluid strata. References to this paper are given in the form (Bjerknes . . .).

Introducing with Professor Bjerknes (Bjerknes p. 18) the quantities Γ, viz:

(5) $\Gamma_1 = (a_2^{\mathrm{I}} - a_2^{\mathrm{II}})(p_2 - p_1)$, $\quad \Gamma_2 = (a_3^{\mathrm{II}} - a_3^{\mathrm{III}})(p_3 - p_1)$, $\quad \Gamma_3 = a_4^{\mathrm{III}}(p_4 - p_1)$,

and further as $p_4 > p_3 > p_2 > p_1$:

(6) $$\pi_{32} = \frac{p_3 - p_2}{p_3 - p_1} < 1, \qquad \pi_{42} = \frac{p_4 - p_2}{p_4 - p_1} < 1, \qquad \pi_{43} = \frac{p_4 - p_3}{p_4 - p_1} < 1$$

we bring the determinant to the simpler form

(7) $$\begin{vmatrix} \Gamma_3 - c^2 & \Gamma_3(1 - \pi_{43}) & \Gamma_3(1 - \pi_{42}) \\ \Gamma_2 & \Gamma_2 - c^2 & \Gamma_2(1 - \pi_{32}) \\ \Gamma_1 & \Gamma_1 & \Gamma_1 - c^2 \end{vmatrix} = 0.$$

Or developed

(8) $$c^6 - (\Gamma_1 + \Gamma_2 + \Gamma_3)c^4 + (\pi_{32}\Gamma_1\Gamma_2 + \pi_{42}\Gamma_1\Gamma_3 + \pi_{43}\Gamma_2\Gamma_3)c^2 - \pi_{43}\pi_{32}\Gamma_1\Gamma_2\Gamma_3 = 0.$$

To convert this into a form which is still more convenient for our purpose, we introduce

$$\varepsilon_1 = \frac{\Gamma_1}{\Gamma_3}, \qquad \varepsilon_2 = \frac{\Gamma_2}{\Gamma_3}$$

and get

(9) $$c^6 - \Gamma_3(1 + \varepsilon_1 + \varepsilon_2)c^4 + \Gamma_3^2(\pi_{32}\varepsilon_1\varepsilon_2 + \pi_{42}\varepsilon_1 + \pi_{43}\varepsilon_2)c^2 - \pi_{43}\pi_{32}\Gamma_3^3\varepsilon_1\varepsilon_2 = 0.$$

Writing this cubic equation in the form

$$x^3 + Ax^2 + Bx + C = 0,$$

we have the coefficients

$$A = -\Gamma_3(1 + \varepsilon_1 + \varepsilon_2), \quad B = \Gamma_3^2(\pi_{32}\varepsilon_1\varepsilon_2 + \pi_{42}\varepsilon_1 + \pi_{43}\varepsilon_2), \quad C = -\pi_{43}\pi_{32}\Gamma_3^3\varepsilon_1\varepsilon_2.$$

As usual in the solution of an equation of the third degree, we introduce the quantities

$$a = B - \frac{1}{3}A^2, \qquad b = -\frac{2}{27}A^3 + \frac{1}{3}AB - C,$$

or introducing the values of A, B, C

$$a = -\frac{1}{3}\Gamma_3^2\left[1 + \varepsilon_1 + \varepsilon_2)^2 - 3(\pi_{32}\varepsilon_1\varepsilon_2 + \pi_{42}\varepsilon_1 + \pi_{43}\varepsilon_2)\right],$$

$$b = \frac{1}{27}\Gamma_3^3\left[2(1 + \varepsilon_1 + \varepsilon_2)^3 - 9(1 + \varepsilon_1 + \varepsilon_2)(\pi_{32}\varepsilon_1\varepsilon_2 + \pi_{42}\varepsilon_1 + \pi_{43}\varepsilon_2) + 27\pi_{43}\pi_{32}\varepsilon_1\varepsilon_2\right].$$

If we consider ε_1 and ε_2 as small compared with Γ_3, i. e. the differences of volumes $a_2^{\mathrm{I}} - a_2^{\mathrm{II}}$ and $a_3^{\mathrm{II}} - a_3^{\mathrm{III}}$ small compared with the volume a_4^{III}, a will be negative, and the equation will have three real roots.

In this case we use the expressions

$$R = \frac{a^3}{27} + \frac{b^2}{4}, \quad \sin\varphi = \sqrt{\frac{27R}{a^3}}, \quad \cos\varphi = \frac{b}{2}\sqrt{-\frac{27}{a^3}}.$$

In the expression of R we leave out terms which are of the second order in ε. Then we get

$$R = -\frac{1}{27} \Gamma_3^6 \left[\frac{1}{4} (\pi_{42} \varepsilon_1 + \pi_{43} \varepsilon_2)^2 - \pi_{43} \pi_{32} \varepsilon_1 \varepsilon_2 \right] ,$$

and further

$$\sin \varphi = \sqrt{\frac{27}{4} [(\pi_{42} \varepsilon_1 + \pi_{43} \varepsilon_2)^2 - 4 \pi_{43} \pi_{32} \varepsilon_1 \varepsilon_2]} , \qquad \cos \varphi = 1 .$$

As the quantities ε are small, we can substitute the argument for the sinus, $\varphi = \sin \varphi$. As is known, the three roots are then

$$x_1 = 2 \sqrt{-\frac{a}{3}} \cos \frac{\varphi}{3} - \frac{1}{3} A ,$$

$$x_2 = -2 \sqrt{-\frac{a}{3}} \cos \left(\frac{\pi}{3} + \frac{\varphi}{3} \right) - \frac{1}{3} A ,$$

$$x_3 = -2 \sqrt{-\frac{a}{3}} \cos \left(\frac{\pi}{3} - \frac{\varphi}{3} \right) - \frac{1}{3} A .$$

As the argument is small, we have

$$\cos \frac{\varphi}{3} = 1 , \quad \cos \left(\frac{\pi}{3} \pm \frac{\varphi}{3} \right) = \frac{1}{2} \mp \frac{1}{2 \sqrt{3}} \varphi .$$

If we neglect second order terms in ε, this gives us

$$x_1 = \Gamma_3 [1 + (1 - \pi_{42}) \varepsilon_1 + (1 - \pi_{43}) \varepsilon_2] ,$$

$$x_2 = \frac{1}{2} \Gamma_3 [\pi_{42} \varepsilon_1 + \pi_{43} \varepsilon_2 + \sqrt{(\pi_{42} \varepsilon_1 + \pi_{43} \varepsilon_2)^2 - 4 \pi_{43} \pi_{32} \varepsilon_1 \varepsilon_2}] ,$$

$$x_3 = \frac{1}{2} \Gamma_3 [\pi_{42} \varepsilon_1 + \pi_{43} \varepsilon_2 - \sqrt{(\pi_{42} \varepsilon_1 + \pi_{43} \varepsilon_2)^2 - 4 \pi_{43} \pi_{32} \varepsilon_1 \varepsilon_2}] .$$

These expressions may also be written in the following form, if instead of x_1, x_2, x_3 we introduce the squares of the velocities of propagation c_1^2, c_2^2, c_3^2:

$$(11) \quad \begin{aligned} c_1^2 &= \Gamma_3 + (1 - \pi_{43}) \Gamma_2 + (1 - \pi_{42}) \Gamma_1 , \\ c_2^2 &= \frac{\pi_{42} \Gamma_1 + \pi_{43} \Gamma_2}{2} + \frac{1}{2} \sqrt{(\pi_{42} \Gamma_1 + \pi_{43} \Gamma_2)^2 - 4 \pi_{43} \pi_{32} \Gamma_1 \Gamma_2} , \\ c_3^2 &= \frac{\pi_{42} \Gamma_1 + \pi_{43} \Gamma_2}{2} - \frac{1}{2} \sqrt{(\pi_{42} \Gamma_1 + \pi_{43} \Gamma_2)^2 - 4 \pi_{43} \pi_{32} \Gamma_1 \Gamma_2} . \end{aligned}$$

A simple transformation gives

$$(12) \quad \begin{aligned} c_2^2 &= \frac{\pi_{42} \Gamma_1 + \pi_{43} \Gamma_2}{2} + \frac{1}{2} \sqrt{(\pi_{42} \Gamma_1 - \pi_{43} \Gamma_2)^2 + 4 \pi_{43} (\pi_{42} - \pi_{32}) \Gamma_1 \Gamma_2} , \\ c_3^2 &= \frac{\pi_{42} \Gamma_1 + \pi_{43} \Gamma_2}{2} - \frac{1}{2} \sqrt{(\pi_{42} \Gamma_1 - \pi_{43} \Gamma_2)^2 + 4 \pi_{43} (\pi_{42} - \pi_{32}) \Gamma_1 \Gamma_2} . \end{aligned}$$

Since the expression under the radical sign is positive, we must have

$$(\pi_{42}\,\Gamma_1 + \pi_{43}\,\Gamma_2)^2 \geqq 4\,\pi_{43}\,\pi_{32}\,\Gamma_1\,\Gamma_2\ ,$$

and obtain the two inequalities

$$(\pi_{42}\,\Gamma_1 + \pi_{43}\,\Gamma_2) \geqq c_2^2 > \frac{\pi_{42}\,\Gamma_1 + \pi_{43}\,\Gamma_2}{2}\ ,$$

$$\frac{\pi_{42}\,\Gamma_1 + \pi_{43}\,\Gamma_2}{2} > c_3^2 \geqq 0\ .$$

We will now determine the horizontal and the vertical amplitudes which correspond to the different velocities of propagation c. For the horizontal amplitudes A we have

$$(13) \qquad \frac{A^{\mathrm{I}}}{\triangle_1} = \frac{A^{\mathrm{II}}}{\triangle_2} = \frac{A^{\mathrm{III}}}{\triangle_3}\ ,$$

$\triangle$ being underdeterminants of the determinant (3). Further, for the vertical amplitudes B:

$$(14) \qquad \begin{aligned} B^{\mathrm{I}} &= -\frac{1}{g}\left[a_1\,(p_2 - p_1)\,A^{\mathrm{I}} + a_2\,(p_3 - p_2)\,A^{\mathrm{II}} + a_3\,(p_4 - p_3)\,A^{\mathrm{III}}\right],\\ B^{\mathrm{II}} &= -\frac{1}{g}\left[(a_2 - a_2^{\mathrm{II}})\,(p_2 - p_1)\,A^{\mathrm{I}} + a_2\,(p_3 - p_2)\,A^{\mathrm{II}} + a_3\,(p_4 - p_3)\,A^{\mathrm{III}}\right],\\ B^{\mathrm{III}} &= -\frac{1}{g}\left[(a_3 - a_3^{\mathrm{III}})\,(p_2 - p_1)\,A^{\mathrm{I}} + (a_3 - a_3^{\mathrm{III}})\,(p_3 - p_2)\,A^{\mathrm{II}} + a_3\,(p_4 - p_3)\,A^{\mathrm{III}}\right] \end{aligned}$$

a being volume quantities defined in Professor Bjerknes' paper (p. 13):

$$a_1 = a_2^{\mathrm{I}} - a_2^{\mathrm{II}} + a_3^{\mathrm{II}} - a_3^{\mathrm{III}} + a_4^{\mathrm{III}}\ , \quad a_2 = a_3^{\mathrm{II}} - a_3^{\mathrm{III}} + a_4^{\mathrm{III}}\ , \quad a_3 = a_4^{\mathrm{III}}\ .$$

Eliminating A^{I} and A^{II} by (13) and leaving out terms of the order of magnitude ε we get

$$(15) \qquad \begin{aligned} B^{\mathrm{I}} &= -\frac{a_4^{\mathrm{III}}}{g\,\triangle_3}\left[(p_2 - p_1)\,\triangle_1 + (p_3 - p_2)\,\triangle_2 + (p_4 - p_3)\,\triangle_3\right] A^{\mathrm{III}},\\ B^{\mathrm{II}} &= -\frac{a_4^{\mathrm{III}}}{g\,\triangle_3}\left[\left(1 - \frac{a_2^{\mathrm{II}}}{a_4^{\mathrm{III}}}\right)(p_2 - p_1)\,\triangle_1 + (p_3 - p_2)\,\triangle_2 + (p_4 - p_3)\,\triangle_3\right] A^{\mathrm{III}},\\ B^{\mathrm{III}} &= -\frac{a_4^{\mathrm{III}}}{g\,\triangle_3}\left[\left(1 - \frac{a_3^{\mathrm{III}}}{a_4^{\mathrm{III}}}\right)(p_2 - p_1)\,\triangle_1 + \right.\\ &\qquad \left. + \left(1 - \frac{a_3^{\mathrm{III}}}{a_4^{\mathrm{III}}}\right)(p_3 - p_2)\,\triangle_2 + (p_4 - p_3)\,\triangle_3\right] A^{\mathrm{III}}. \end{aligned}$$

From the matrix

$$(16) \qquad \left|\begin{matrix} [\Gamma_1 + (1 - \pi_{32})\,\Gamma_2 + (1 - \pi_{42})\,\Gamma_3] - c^2 & \pi_{32}\,\Gamma_2 + (\pi_{42} - \pi_{43})\,\Gamma_3 & \pi_{43}\,\Gamma_3 \\ (1 - \pi_{32})\,\Gamma_2 + (1 - \pi_{42})\,\Gamma_3 & [\pi_{32}\,\Gamma_2 + (\pi_{42} - \pi_{43})\,\Gamma_3] - c^2 & \pi_{43}\,\Gamma_3 \end{matrix}\right|$$

we form the three determinants:

$$(17) \qquad \begin{aligned} \triangle_1 &= \pi_{43}\,\Gamma_3\,c^2\ ,\\ \triangle_2 &= \pi_{43}\,(c^2 - \Gamma_1)\,\Gamma_3\ ,\\ \triangle_3 &= c^4 - [\Gamma_1 + \Gamma_2 + (1 - \pi_{43})\,\Gamma_3]\,c^2 + \Gamma_1\,[\pi_{32}\,\Gamma_2 + (\pi_{42} - \pi_{43})\,\Gamma_3\ . \end{aligned}$$

Substituting the greatest root c^2, and leaving out quantities of the magnitude ε, we get

$$\triangle_1 = \triangle_2 = \triangle_3 = \pi_{43} \Gamma_3'^2 > 0 . \tag{18}$$

Thus in the greatest velocity of propagation we have motion of the same direction in all strata. If we suppose A^{III} to be positive we get under the same suppositions:

$$\begin{aligned} B^{I} &= -\frac{1}{g} \Gamma_3' A^{III} < 0 . \\ B^{II} &= -\frac{1}{g} [\Gamma_3' - (p_2 - p_1) a_2^{II}] A^{III} \lessgtr 0 , \text{ according as } \Gamma_3' \gtrless (p_2 - p_1) a_2^{II}. \\ B^{III} &= -\frac{1}{g} [\Gamma_3' - (p_3 - p_1) a_3^{III}] A^{III} \lessgtr 0 , \text{ according as } \Gamma_3' \gtrless (p_3 - p_1) a_3^{III}. \end{aligned} \tag{19}$$

If we wish to substitute the two smaller roots (11) or (12) it will be sufficient to consider the case when the quantity $\pi_{43} \pi_{32} \Gamma_1 \Gamma_2$ is very near zero. Then c_2^2 may be written

$$c_2^2 = \pi_{42} \Gamma_1' + \pi_{43} \Gamma_2' - \frac{\pi_{43} \pi_{32} \Gamma_1' \Gamma_2'}{\pi_{42} \Gamma_1' + \pi_{43} \Gamma_2'} \tag{20}$$

Retaining only such terms as do not contain ε we then get the determinants:

$$\begin{aligned} \triangle_1 &= \pi_{43} (\pi_{42} \Gamma_1' + \pi_{43} \Gamma_2') \Gamma_3' > 0 . \\ \triangle_2 &= \pi_{43} [\pi_{43} \Gamma_2' - (1 - \pi_{42}) \Gamma_1'] \Gamma_3' \gtrless 0 , \text{ according as } \frac{\Gamma_2'}{\Gamma_1'} \gtrless \frac{1 - \pi_{42}}{\pi_{43}} . \\ \triangle_3 &= \pi_{43} [(1 - \pi_{42}) \Gamma_1' + (1 - \pi_{43}) \Gamma_2'] \Gamma_3' < 0 . \end{aligned} \tag{21}$$

Thus two adjacent strata have the same horizontal motion and the third the opposite motion: according to the differences of pressure, which determine the content of mass in each stratum, the intermediate stratum will either follow the upper or the lower stratum in its motion.

For a determination of the vertical amplitudes we must retain the quantities ε and we get for both c_2^2 and c_3^2

$$\begin{aligned} B^{I} &= -\frac{1}{g \triangle_3} \pi_{43} \Gamma_3' c^4 A^{III}. \\ B^{II} &= -\frac{1}{g \triangle_3} \pi_{43} \Gamma_3' c^2 [c^2 - (p_2 - p_1) a_2^{I}] A^{III}. \\ B^{III} &= -\frac{1}{g \triangle_3} \pi_{43} \Gamma_3' [c^2 (c^2 - \Gamma_1') - (p_3 - p_1) (c^2 - \pi_{32} \Gamma_1') a_3^{II} A^{III}. \end{aligned} \tag{22}$$

If we consider the volumes a_2^{I} and a_3^{II} as large compared with the differences in volume, this will give for the intermediate velocity c_2:

$$\begin{aligned} B^{I} &= -\frac{1}{g \triangle_3} \pi_{43} \Gamma_3' (\pi_{42} \Gamma_1' + \pi_{43} \Gamma_2')^2 A^{III} > 0 . \\ B^{II} &= \frac{1}{g \triangle_3} \pi_{43} \Gamma_3' (\pi_{42} \Gamma_1' + \pi_{43} \Gamma_2') (p_2 - p_1) a_2^{I} A^{III} < 0 . \\ B^{III} &= \frac{1}{g \triangle_3} \pi_{43} \Gamma_3' [\pi_{43} (1 - \pi_{32}) \Gamma_1' + \pi_{43} \Gamma_2'] (p_3 - p_1) a_3^{II} A^{III} < 0 \cdot \end{aligned} \tag{23}$$

For the smallest velocity

$$c_3^2 = \frac{\pi_{43}\,\pi_{32}\,\Gamma_1\,\Gamma_2}{\pi_{42}\,\Gamma_1 + \pi_{43}\,\Gamma_2} \tag{24}$$

the horizontal amplitudes will be determined by the three determinants

$$\begin{aligned} \triangle_1 &= \frac{\pi_{43}^2\,\pi_{32}\,\Gamma_1\,\Gamma_2}{\pi_{42}\,\Gamma_1 + \pi_{43}\,\Gamma_2}\,\Gamma_3 > 0\,. \\ \triangle_2 &= -\,\pi_{43}\,\frac{\pi_{42}\,\Gamma_1 + \pi_{43}\,(1 - \pi_{32})\,\Gamma_2}{\pi_{42}\,\Gamma_1 + \pi_{43}\,\Gamma_2}\,\Gamma_1\,\Gamma_3 < 0\,. \\ \triangle_3 &= \frac{\pi_{43}^2\,\pi_{32}\,\Gamma_2 + \pi_{42}\,(\pi_{42} - \pi_{43})\,\Gamma_1}{\pi_{42}\,\Gamma_1 + \pi_{43}\,\Gamma_2}\,\Gamma_1\,\Gamma_3 > 0\,. \end{aligned} \tag{25}$$

Substituting this in (22) and remembering that c_3^2 is very smale, we get

$$\begin{aligned} B^{\mathrm{I}} &= -\frac{1}{g\,\triangle_3}\,\pi_{43}\,\Gamma_3\,c_3^4\,A^{\mathrm{III}} < 0 \\ B^{\mathrm{II}} &= \frac{1}{g\,\triangle_3}\,\pi_{43}\,\Gamma_3\,c_3^2\,(p_2 - p_1)\,\alpha_2^{\mathrm{I}}\,A^{\mathrm{III}} > 0 \\ B^{\mathrm{III}} &= -\frac{1}{g\,\triangle_3}\,\pi_{43}\,\pi_{32}\,\Gamma_1\,\Gamma_3\,(p_3 - p_1)\,\alpha_3^{\mathrm{II}}\,A^{\mathrm{III}} < 0\,, \end{aligned} \tag{26}$$

which shows that the upper and the lower stratum have the same direction of motion, but the intermediate stratum the opposite direction.

It should be observed that this is demonstrated only for the case when c_2 is near its maximum and c_3 near its minimum value. But if $4\,\pi_{43}\,\pi_{32}\,\Gamma_1\,\Gamma_2$ is of the same order of magnitude as $\pi_{42}\,\Gamma_1 + \pi_{43}\,\Gamma_2$ we have other relations between the amplitudes.

As an example of the applications of the preceding formulae to atmospheric motions, we may introduce the gas equation

$$p\,\alpha = R\,\vartheta$$

R being the gas constant and ϑ the absolute temperature. To simplify we put $p_1 = 0$ which gives

$$\Gamma_1 = R\,(\vartheta_2^{\mathrm{I}} - \vartheta_2^{\mathrm{II}})\,, \qquad \Gamma_2 = R\,(\vartheta_3^{\mathrm{II}} - \vartheta_3^{\mathrm{III}})\,, \qquad \Gamma_3 = R\,\vartheta_4^{\mathrm{III}}\,, \tag{27}$$

and introducing the pressures p instead of the auxiliary quantities π, we get for the three squared velocities:

$$\begin{aligned} c_1^2 &= R\,\vartheta_4^{\mathrm{III}} + \frac{R}{p_4}\,[p_3\,(\vartheta_3^{\mathrm{II}} - \vartheta_3^{\mathrm{III}}) + p_2\,(\vartheta_2^{\mathrm{I}} - \vartheta_2^{\mathrm{II}})]\,, \\ c_{2,3}^2 &= \frac{R}{2\,p_4}\,[(p_4 - p_2)\,(\vartheta_2^{\mathrm{I}} - \vartheta_2^{\mathrm{II}}) + (p_4 - p_3)\,(\vartheta_3^{\mathrm{II}} - \vartheta_3^{\mathrm{III}})] \pm \\ &\pm \frac{R}{2\,p_4}\sqrt{\begin{aligned}&[(p_4 - p_2)\,(\vartheta_2^{\mathrm{I}} - \vartheta_2^{\mathrm{II}}) + (p_4 - p_3)\,(\vartheta_3^{\mathrm{II}} - \vartheta_3^{\mathrm{III}})]^2 - \\ &- 4\,(p_4 - p_3)\,(p_3 - p_2)\,\frac{p_4}{p_3}\,(\vartheta_2^{\mathrm{I}} - \vartheta_2^{\mathrm{II}})\,(\vartheta_3^{\mathrm{II}} - \vartheta_3^{\mathrm{III}})\,.\end{aligned}} \end{aligned} \tag{28}$$

Beginning with the greatest velocity c_1 and remembering that in the adiabatic temperature distribution within each stratum we have $\vartheta_4^{III} > \vartheta_3^{III}$ and $\vartheta_3^{II} > \vartheta_2^{II}$, we obtain

(29)
$$\begin{aligned} A^{I} &> 0, & B^{I} &= -\frac{R}{g}\vartheta_4^{III} A^{III} < 0 . \\ A^{II} &> 0, & B^{II} &= -\frac{R}{g}(\vartheta_4^{III} - \vartheta_2^{II}) A^{III} < 0 . \\ A^{III} &> 0, & B^{III} &= -\frac{R}{g}(\vartheta_4^{III} - \vartheta_3^{III}) A^{III} < 0 . \end{aligned}$$

These formulae show that both the horizontal and the vertical amplitudes have equal direction, and the entire system practically moves like a single stratum, having at the free surface a vertical amplitude depending only upon A^{III} and the temperature ϑ_4^{III} at the ground:

$$B = -\frac{R}{g}\vartheta_4^{III} A^{III}.$$

For the limiting value of c_2 we have

(30)
$$c_2^2 = \frac{R}{p_4}[(p_4 - p_2)(\vartheta_2^{I} - \vartheta_2^{II}) + (p_4 - p_3)(\vartheta_3^{II} - \vartheta_3^{III})] - \\ - R\frac{(p_4 - p_3)}{p_3}\cdot\frac{(p_3 - p_2)(\vartheta_2^{I} - \vartheta_2^{II})(\vartheta_3^{II} - \vartheta_3^{III})}{(p_4 - p_2)(\vartheta_2^{I} - \vartheta_2^{II}) + (p_4 - p_3)(\vartheta_3^{II} - \vartheta_3^{III})},$$

and get

(31)
$$\begin{aligned} A^{I} &> 0, & & & B^{I} &> 0 . \\ A^{II} &\gtrless 0, & &\text{when } \frac{\vartheta_3^{II} - \vartheta_3^{III}}{\vartheta_2^{I} - \vartheta_2^{II}} \gtrless \frac{p_2}{p_4 - p_3}, & B^{II} &< 0 . \\ A^{III} &< 0, & & & B^{III} &< 0 . \end{aligned}$$

When the smallest velocity c_3,

(32)
$$c_3^2 = \frac{R}{p_3}\cdot\frac{(p_4 - p_3)(p_3 - p_2)(\vartheta_2^{I} - \vartheta_2^{II})(\vartheta_3^{II} - \vartheta_3^{III})}{(p_4 - p_2)(\vartheta_2^{I} - \vartheta_2^{II}) + (p_4 - p_3)(\vartheta_3^{II} - \vartheta_3^{III})},$$

is very near zero, we get correspondingly:

(33)
$$\begin{aligned} A^{I} &> 0, & B^{I} &< 0 . \\ A^{II} &< 0, & B^{II} &> 0 . \\ A^{III} &> 0, & B^{III} &< 0 . \end{aligned}$$

As pointed out at the end of Professor Bjerknes' paper, it may be of considerable interest to examine the approximation with which a system of two strata may be substituted for one of three strata. We denote by $\bar{c}$, $\bar{\pi}$, $\bar{\Gamma}$ etc. the symbols for the system of two strata which is artificially introduced instead of that of three strata, and we limit ourselves to a comparison of the intermediate velocity of propagation for three strata with the smallest for two strata. Starting with a system of three strata and trying to determine one of two strata having the same velocity of propagation $\bar{c}_2 = c_2$, we may dispose of the pressure $\bar{p}$

(34)
$$\bar{p}_2 = \frac{p_3 + p_1}{2}$$

and determine the required discontinuity of volume for the system of two strata. As we have $\bar{p}_3 - \bar{p}_1 = p_4 - p_1$ we find

$$(35) \qquad 2\,\bar{\pi}_{32}\,\bar{\Gamma}_1 = \pi_{42}\,\Gamma_1 + \pi_{43}\,\Gamma_2 + \sqrt{(\pi_{42}\,\Gamma_1 + \pi_{43}\,\Gamma_2)^2 - 4\,\pi_{43}\,\pi_{32}\,\Gamma_1\,\Gamma_2}$$

or according to (34)

$$(36) \qquad \left(p_4 - \frac{p_3 + p_2}{2}\right)(\bar{\alpha}_2^{\mathrm{I}} - \bar{\alpha}_2^{\mathrm{II}}) = \frac{1}{\pi_{42} + \pi_{43}}\,[\pi_{42}\,\Gamma_1 + \pi_{43}\,\Gamma_2 + \\ + \sqrt{(\pi_{42}\,\Gamma_1 + \pi_{43}\,\Gamma_2)^2 - 4\,\pi_{43}\,\pi_{32}\,\Gamma_1\,\Gamma_2}\,],$$

which gives the solution of our problem. It is immediately seen from this formula, that if the intermediate stratum shrinks to zero, so that $p_3 = p_2$, we get the identity

$$\bar{\Gamma}_1 = \Gamma_1 + \Gamma_2\,, \quad \text{or } \bar{\alpha}_2^{\mathrm{I}} - \bar{\alpha}_2^{\mathrm{II}} = \alpha_2^{\mathrm{I}} - \alpha_2^{\mathrm{III}}.$$

If vice versa we pass from three to two strata and suppose $\Gamma_1 = \Gamma_2$, we get

$$(37) \qquad \Gamma_1 = \Gamma_2 = \frac{\bar{\pi}_{32}\,\bar{\Gamma}_1}{2\,\pi_{43}\,\pi_{32}}\,[-(\pi_{42} + \pi_{43}) + \sqrt{(\pi_{42} + \pi_{43})^2 + 4\,\pi_{43}\,\pi_{32}}\,].$$

Disposing here again of the pressures in accordance with the condition (34) and putting $\bar{\pi}_{31} = \dfrac{p_3 - \bar{p}_1}{p_3 - p_1}$, we find

$$(38) \qquad (p_4 + p_3 - 2\,\bar{p}_2)(\alpha_2^{\mathrm{I}} - \alpha_2^{\mathrm{II}}) = (p_4 - p_3)(\alpha_3^{\mathrm{II}} - \alpha_3^{\mathrm{III}}) = \\ = \frac{\bar{\pi}_{32}}{2\,\pi_{43}\,\bar{\pi}_{31}}\,\bar{\Gamma}_1\,[-\bar{\pi}_{32} + \sqrt{\bar{\pi}_{32}^{\,2} + 2\,\pi_{43}\,\bar{\pi}_{31}}\,].$$

Retaining in this expression a suitable value of p_3, i. e. a value fulfilling the condition $p_4 > p_3 > \bar{p}_2$, we get a definite value of the volume differences. For $p_3 = p_2 = \bar{p}_2$ the expression takes indeterminate form. But the regular method gives for this case

$$\alpha_2^{\mathrm{I}} - \alpha_2^{\mathrm{III}} = \bar{\alpha}_2^{\mathrm{I}}\,\bar{\alpha}_2^{\mathrm{II}}\,,$$

just as above.

The two following tables giving the velocities c_2 and c_3 are calculated from the formula (28). As pressures defining the boundary surfaces we have chosen

$$p_4 = 100 \text{ cbar}, \quad p_3 = 70 \text{ cbar}, \quad p_2 = 30 \text{ cbar}, \quad p_1 = 0\,,$$

p_4 being the normal pressure at sea level, p_3 representing the pressure at a certain average height of the polar front surface, and p_2 the pressure at the boundary between troposphere and stratosphere. In the table for the velocity c_2 two different kinds of type have been used: the italics are used when the two upper strata have the same direction of motion, the ordinary type when the two lower strata have the same direction of motion.

c_2 m/sec.

$\vartheta_2^{I} - \vartheta_2^{II}$ C°	$\vartheta_3^{II} - \vartheta_3^{III}$ C°											
	0.2	0.4	0.6	0.8	1	2	4	6	8	10	15	20
0.2	7	7	*8*	*9*	*10*	*13*	*19*	*23*	*26*	*29*	*36*	*42*
0,4	9	9	10	10	11	*14*	*19*	*23*	*26*	*29*	*36*	*42*
0,6	11	11	12	12	12	*15*	*19*	*23*	*26*	*29*	*36*	*42*
0,8	13	13	13	13	14	*15*	*20*	*24*	*27*	*30*	*36*	*42*
1	14	14	15	15	15	16	*20*	*24*	*27*	*30*	*37*	*42*
2	20	20	20	20	21	21	23	*26*	*29*	*31*	*37*	*43*
4	28	28	28	29	29	29	30	31	33	*35*	*39*	*44*
6	35	35	35	35	35	35	36	37	38	39	*42*	*46*
8	40	40	40	40	40	41	41	42	42	43	46	*49*
10	45	45	45	45	45	45	46	46	47	47	49	52
15	55	55	55	55	55	55	56	56	56	57	58	60
20	63	63	63	63	63	64	64	64	64	65	66	67

c_3 m/sec.

$\vartheta_2^{I} - \vartheta_2^{II}$ C°	$\vartheta_3^{II} - \vartheta_3^{III}$ C°											
	0.2	0.4	0.6	0.8	1	2	4	6	8	10	15	20
0.2	4	5	5	5	5	6	6	6	6	6	6	6
0.4	4	5	6	6	7	8	8	8	8	8	8	8
0.6	4	5	6	7	7	9	10	10	10	10	10	10
0.8	4	5	6	7	8	10	11	11	11	11	11	11
1	4	5	6	7	8	10	12	12	12	13	13	13
2	4	5	6	7	8	11	15	16	17	17	18	18
4	4	5	6	7	8	12	16	19	21	22	23	24
6	4	5	6	7	8	12	16	19	22	24	27	28
8	4	5	6	7	8	12	16	20	22	25	29	31
10	4	5	6	7	8	12	16	20	23	25	30	33
15	4	5	6	7	8	12	17	20	23	26	31	35
20	4	5	6	7	8	12	17	20	23	26	31	36