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On the Stability of Shear Flow of a Stratified Fluid

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Here N is a positive integer and Θ is defined by

(2.59) $$\Theta=\pi(\tfrac{1}{2}-\sqrt{\tfrac{1}{4}-r_0})\,.$$

The solutions of equation (2.56) for values of γ satisfying the conditions (2.58) are

(2.60) $$j=\frac{1}{\ln(4\gamma^2)}\left(\ln\frac{\sin 2\gamma}{\sin\Theta}+(2n+1)\pi i\right) \quad \text{and}$$
$$j=\frac{1}{\ln(4\gamma^2)}\left(\ln\frac{|\sin 2\gamma|}{\sin\Theta}+2n\pi i\right)$$

respectively, where n is an integer or zero. These solutions are seen to be good approximations to the solutions of equation (2.39) if $|j|\ll 1$, i.e. if

(2.61) $$\frac{|n|}{\ln\gamma}\ll 1\,.$$

For a given large value of γ, only a finite number of the solutions (2.60) can be accepted.

The values of c corresponding to the above solutions are found from the definition equation (2.38) of j. It obtains

(2.62) $$c=\frac{\gamma}{(\gamma^2+k^2)\ln(4\gamma^2)}\left(-(2n+1)\pi+i\ln\frac{\sin 2\gamma}{\sin\Theta}\right) \quad \text{and}$$
$$c=\frac{\gamma}{(\gamma^2+k^2)\ln(4\gamma^2)}\left(-2n\pi+i\ln\frac{|\sin 2\gamma|}{\sin\Theta}\right).$$

From these expressions for c the transition to instability waves should occur when

(2.63) $$|\sin 2\gamma|=\sin\Theta\,.$$

That means that in the considered interval (2.58) all contingent instability waves will appear for the same value of 2γ. This is not correct. The next approximation, which will not be included in our discussion, will show that the transition takes place principally in the same manner as for $r_0=0$, i.e. for a discrete set of γ-values. In the case here considered all instability waves originate from singular solutions. No special solution exists.

For interpreting our results in the k-space we will draw attention to the fact that decreasing values of k correspond to increasing values of γ.

For γ given by

$$\frac{1}{\pi}\ln(4\gamma^2)\sin 2\gamma = \frac{\pi}{2}, \tag{2.53}$$

which corresponds to the transition between the region where no instability wave occurs and the region where we get one instability wave with $c_r \neq 0$, it is easily verified that $\xi = 0$, i.e. $c_i = 0$, is a solution of equations (2.46). This transition value of γ, therefore, represents a solution with singularity (singular solution).

A new instability wave originating from a singular solution will occur each time γ passes the values given by

$$\frac{1}{\pi}\ln(4\gamma^2)\sin 2\gamma = (2q+\tfrac{1}{2})\pi, \tag{2.54}$$

with q an integer. This continues until $\frac{1}{\pi}\ln(4\gamma^2)\sin 2\gamma$ reaches its maximum value at 2γ approximately equal to $(2N+\frac{1}{2})\pi$. As γ then increases further, $\ln(4\gamma^2)\sin 2\gamma$ will decrease. The number of instability waves occurring will decrease by one each time γ passes the value corresponding to a singular solution. Again for 2γ slightly less than $(2N+1)\pi$ we have no instability solution. When

$$2\gamma = (2N+1)\pi, \tag{2.55}$$

a special solution will occur. Further increase of γ leads to a repetition of appearance of new instability waves with $c_r \neq 0$ in the same manner as explained above. In addition we now also have an instability wave with $c_r = 0$.

$\underline{0 < r_0 < \frac{1}{4}}$

In this case we put $c = 0$ as a first approximation of eq. (2.39). Putting $c = 0$ on the right hand side of the equation, we obtain as a next approximation

$$e^{j\ln 4\gamma^2} = -\frac{\sin 2\gamma}{\sin \pi(\frac{1}{2}-\sqrt{\frac{1}{4}-r_0})}. \tag{2.56}$$

This equations has solutions with $c_i > 0$ only when

$$\left|\frac{\sin 2\gamma}{\sin \pi(\frac{1}{2}-\sqrt{\frac{1}{4}-r_0})}\right| > 1. \tag{2.57}$$

We notice that this condition cannot be fulfilled for $r_0 > \frac{1}{4}$. That no instability waves exist for $r_0 > \frac{1}{4}$ is in accordance with a more general result obtained by MILES (1961).

The above condition (2.57) is fulfilled when

$$\begin{gathered}(2N-1)\pi + \Theta < 2\gamma < 2N\pi - \Theta \quad \text{or} \\ 2N\pi + \Theta < 2\gamma < (2N+1)\pi - \Theta.\end{gathered} \tag{2.58}$$

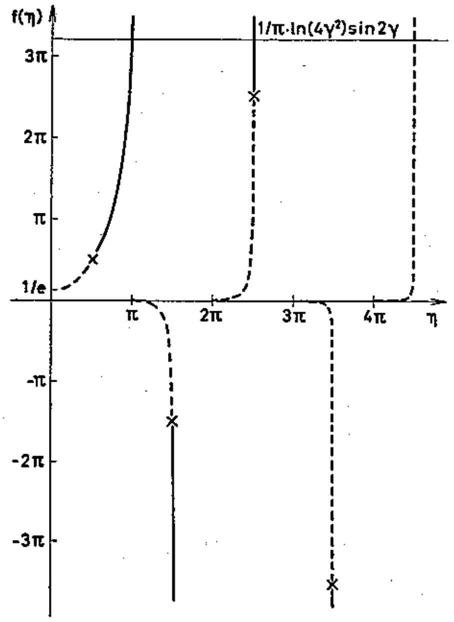

Fig. 1. The solid drawn part of the graphs of $f(\eta)$ represents acceptable solutions of the last of equations (2.46). Choosing, for instance, $\frac{1}{\pi}\ln(4\gamma^2)\sin 2\gamma = 3.25\pi$, we obtain two acceptable solutions, as indicated in the diagram.

(2.49)
$$\left(2p-\frac{3}{2}\right)\pi<\frac{1}{\pi}\ln(4\gamma^2)\sin 2\gamma<(2p+\tfrac{1}{2})\pi \quad \text{or}$$
$$-\left(2p+\frac{3}{2}\right)\pi<\frac{1}{\pi}\ln(4\gamma^2)\sin 2\gamma<-(2p-\tfrac{1}{2})\pi.$$

Here p, of course is a positive integer or zero.

We may now discuss the solutions for arbitrary values of γ. Choose a positive integer N and consider γ in the region given by $2N\pi \leq 2\gamma \leq (2N+2)\pi$. For $2\gamma = 2N\pi$ we have the special solution with $c=0$ discussed in section 1. This value of 2γ gives the transition from a region with one instability wave with a velocity of propagation equal to zero for 2γ less than $2N\pi$, to a region with no instability waves with zero velocity of propagation. For the special solution we have

(2.50)
$$\ln(4\gamma^2)\sin 2\gamma=0.$$

As γ increases, $\ln(4\gamma^2)\sin 2\gamma$ will increase. As seen from the relations (2.49), we have no instability wave when

(2.51)
$$\frac{1}{\pi}\ln(4\gamma^2)\sin 2\gamma\leq\frac{\pi}{2}.$$

When γ increases further there is one instability wave with $c_r \neq 0$ when γ is in the region

(2.52)
$$\frac{\pi}{2}<\frac{1}{\pi}\ln(4\gamma^2)\sin 2\gamma\leq\frac{5\pi}{2}.$$

and no real solution for positive real values of j when

$$2N\pi < 2\gamma < (2N+1)\pi . \tag{2.43}$$

Here N is a positive integer.

A positive real value of j gives (see eq. (2.38)) $c_r = 0$ and $c_i > 0$ i.e. an instability wave with a velocity of propagation equal to zero. Thus in the region given by equation (2.42) we have a solution representing an instability wave with a velocity of propagation equal to zero, whereas in the region given by equation (2.43) no instability wave of this kind exists. The transition from one of these regions to the other is given by $j=0$ when $\sin 2\gamma = 0$, i.e. the special solutions discussed in section 1.

In order to find contingent instability waves with a velocity of propagation different from zero we put

$$j \ln 4\gamma^2 = \xi + i\eta , \tag{2.44}$$

in equation (2.41). Separating into real and pure imaginary parts we obtain

$$\begin{aligned} e^{\xi}(\xi \cos\eta - \eta \sin\eta) &= -\frac{1}{\pi}\ln(4\gamma^2)\sin 2\gamma , \\ \eta\cos\eta + \xi\sin\eta &= 0 . \end{aligned} \tag{2.45}$$

From these equations we find

$$\begin{aligned} \xi &= -\eta \operatorname{cotg}\eta , \\ f(\eta) &= \frac{\eta}{\sin\eta} e^{-\eta \operatorname{cotg}\eta} = \frac{1}{\pi}\ln(4\gamma^2)\sin 2\gamma . \end{aligned} \tag{2.46}$$

The latter of the last equations may easily be solved graphically. In the diagram Fig. 1 is shown the behaviour of $f(\eta)$ as a function of η.

From the diagram we conclude that equations (2.46) have an infinite number of solutions. In order to make $c_i > 0$, however, we must have $\xi > 0$. Hence, only solutions with

$$(n+\tfrac{1}{2})\pi < \eta < (n+1)\pi \tag{2.47}$$

are acceptable. Accordingly we have only a finite number of solutions (which satisfy our requirements). Since

$$f\{(n+\tfrac{1}{2})\pi\} = (-1)^n (n+\tfrac{1}{2})\pi \tag{2.48}$$

it is easily seen that we have p acceptable solutions when

$$\frac{2\pi i}{\Gamma(\frac{1}{2}+m+j)\Gamma(\frac{1}{2}-m+j)}(\zeta_1\zeta_2)^j e^{-\frac{1}{2}(\zeta_1+\zeta_2)}+\left(\frac{\zeta_1}{\zeta_2}\right)^j e^{\frac{1}{2}(\zeta_2-\zeta_1)}-\left(\frac{\zeta_2}{\zeta_1}\right)^j e^{-\frac{1}{2}(\zeta_2-\zeta_1)}=0\,. \tag{2.35}$$

Substituting the values of ζ_1 and ζ_2 given by equations (2.31) it obtains

$$\frac{2\pi i\{4\gamma^2(1-c^2)\}^j}{\Gamma(\frac{1}{2}+m+j)\Gamma(\frac{1}{2}-m+j)}e^{2i\gamma c}+e^{-i\pi j}\left(\frac{1+c}{1-c}\right)^j e^{2i\gamma}-e^{i\pi j}\left(\frac{1-c}{1+c}\right)^j e^{-2i\gamma}=0 \tag{2.36}$$

where

$$\gamma^2=r_1-r_0-k^2. \tag{2.37}$$

and, due to the definition (2.8),

$$j=-ic\frac{\gamma^2+k^2}{\gamma}, \qquad m^2=\tfrac{1}{4}-r_0-c^2(\gamma^2+k^2). \tag{2.38}$$

Equation (2.36) may be written

$$e^{j\ln 4\gamma^2}=-\frac{\Gamma(\frac{1}{2}+m+j)\Gamma(\frac{1}{2}-m+j)}{\pi}\sin\left(2\gamma-\pi j-ij\ln\frac{1+c}{1-c}\right). \tag{2.39}$$

Since the asymptotic expansions implied finite values of $|j|$ and $|m|$, it is seen from equations (2.38) that we must have $|c|\ll 1$ when $\gamma\gg 1$. By taking $c=0$ as a first approximation, the next approximation can be found by putting $c=0$ on the right hand side of equation (2.39). This will be carried out later considering the case that $0<r_0<\frac{1}{4}$. We will, however, first discuss the instance that

$r_0=0.$

In this case the right hand side of equation (2.39) tends to infinity as c tends to zero. We therefore write the equation in the equivalent form

$$je^{j\ln 4\gamma^2}=-j\frac{\Gamma(\frac{1}{2}+m+j)\Gamma(\frac{3}{2}-m+j)}{\pi(\frac{1}{2}-m+j)}\sin\left(2\gamma-\pi j-ij\ln\frac{1+c}{1-c}\right). \tag{2.40}$$

We may now choose $c=0$ as a first approximation, and the next approximation may be found by putting $c=0$ on the right hand side of equation (2.40). This approximation will then be given by the equation

$$je^{j\ln 4\gamma^2}=-\frac{1}{\pi}\sin 2\gamma\,. \tag{2.41}$$

We first notice that this equation has a solution with a positive real value of j when

$$(2N-1)\pi<2\gamma<2N\pi, \tag{2.42}$$

Accordingly we may substitute the solutions (2.25) in the frequency equation (2.24). Using furthermore the asymptotic expansions (2.21) and (2.20), and retaining only the first term in (2.20), the frequency equation approximates to

$$-\left(\frac{\zeta_2}{\zeta_1}\right)^j e^{-\frac{1}{2}(\zeta_1-\zeta_2)}+\left(\frac{\zeta_1}{\zeta_2}\right)^j e^{\frac{1}{2}(\zeta_1-\zeta_2)}=0. \tag{2.28}$$

Substituting the values for ζ_1 and ζ_2 from (2.12), this equation may be written as

$$\left(\frac{1-c}{-1-c}\right)^{2j}=e^{4(k^2+r_0-r_1)^{\frac{1}{2}}} \tag{2.29}$$

This equation has no solution satisfying the assumptions made above in order to use our asymptotic expansions.

We will then assume

$$r_1-r_0>k^2 \tag{2.30}$$

The values of ζ_1 and ζ_2 may then be written as

$$\zeta_1=2e^{-i(\pi/2)}(r_1-r_0-k^2)^{\frac{1}{2}}(1+c),\quad \zeta_2=2e^{i(\pi/2)}(r_1-r_0-k^2)^{\frac{1}{2}}(1-c). \tag{2.31}$$

With the same assumption of c as before, we find

$$-\frac{\pi}{2}<\arg\zeta_1<0 \qquad 0<\arg\zeta_2<\frac{\pi}{2}. \tag{2.32}$$

In this case, we notice that $Z_2(\zeta_2)$ cannot be given by the last of equations (2.25). We have to return to equation (2.23) in order to find $Z_2(\zeta)$ given by the Whittaker functions valid in another region. Taking the lower sign in the last of equations (2.22), solving it together with the first of the equations with respect to $M_{j,m}$ and $M_{j,-m}$ and substituting in the last of equations (2.23), we find

$$Z_2(\zeta)=\frac{2\pi i}{\Gamma(\frac{1}{2}+m+j)\Gamma(\frac{1}{2}-m+j)}e^{-i\pi j}W_{j,m}(\zeta)+e^{-2\pi ij}W_{-j,m}(\zeta e^{-i\pi}) \tag{2.33}$$

valid when $0\leq\arg\zeta\leq\pi$.

Substituting equation (2.25) for $Z_1(\zeta_1)$, $Z_2(\zeta_1)$ and $Z_1(\zeta_2)$ and the above equation for $Z_2(\zeta_2)$, the frequency equation (2.24) may be written

$$\frac{2\pi i}{\Gamma(\frac{1}{2}+m+j)\Gamma(\frac{1}{2}-m+j)}W_{j,m}(\zeta_1)W_{j,m}(\zeta_2)+e^{-i\pi j}W_{j,m}(\zeta_1)W_{j,m}(\zeta_2e^{-i\pi}) \tag{2.34}$$
$$-e^{i\pi j}W_{j,m}(\zeta_1e^{i\pi})W_{j,m}(\zeta_2)=0.$$

Again we find an approximation to the frequency equation by use of the asymptotic expansions (2.21) with (2.20) retaining only the first term in the last equation. This approximation may be written

(2.21)
$$W_{j,m}(\zeta)=\zeta^j e^{-\frac{1}{2}\zeta} f_j(\zeta), \qquad |\arg\zeta|\leq\pi,$$
$$W_{-j,m}(\zeta e^{\pm i\pi})=(\zeta e^{\pm i\pi})^{-j} e^{\frac{1}{2}\zeta} f_{-j}(-\zeta), \qquad |\arg(\zeta e^{\pm i\pi})|\leq\pi,$$

The relations between the Whittaker functions and our confluent hypergeometric functions $M_{j,m}$ (formula 2.11) are given by

(2.22)
$$W_{j,m}(\zeta)=\frac{\Gamma(-2m)}{\Gamma(\frac{1}{2}-m-j)}M_{j,m}(\zeta)+\frac{\Gamma(2m)}{\Gamma(\frac{1}{2}+m-j)}M_{j,-m}(\zeta) \quad \text{when } |\arg\zeta|\leq\pi,$$
$$W_{-j,m}(\zeta e^{\pm i\pi})=\frac{\Gamma(-2m)}{\Gamma(\frac{1}{2}-m+j)}M_{j,m}(\zeta)e^{\pm i\pi(\frac{1}{2}+m)}+\frac{\Gamma(2m)}{\Gamma(\frac{1}{2}+m+j)}M_{j,-m}(\zeta)e^{\pm i\pi(\frac{1}{2}-m)}$$
$$\text{when } |\arg(\zeta e^{\pm i\pi})|\leq\pi.$$

As linearly independent solutions of equation (2.9) we now choose

(2.23)
$$Z_1(\zeta)=\frac{\Gamma(-2m)}{\Gamma(\frac{1}{2}-m-j)}M_{j,m}(\zeta)+\frac{\Gamma(2m)}{\Gamma(\frac{1}{2}+m-j)}M_{j,-m}(\zeta)$$
$$Z_2(\zeta)=\frac{\Gamma(-2m)}{\Gamma(\frac{1}{2}-m+j)}e^{i\pi(\frac{1}{2}+m)}M_{j,m}(\zeta)+\frac{\Gamma(2m)}{\Gamma(\frac{1}{2}+m+j)}e^{i\pi(\frac{1}{2}-m)}M_{j,-m}(\zeta).$$

The frequency equations becomes

(2.24)
$$Z_1(\zeta_1)Z_2(\zeta_2)-Z_1(\zeta_2)Z_2(\zeta_1)=0.$$

The values of ζ_1 and ζ_2 are still given by the relations (2.12).

The solutions $Z_1(\zeta)$ and $Z_2(\zeta)$ as defined by equations (2.23) are valid for all values of arg ζ. Introducing, however, the Whittaker functions given by equation (2.21), we find

(2.25)
$$Z_1(\zeta)=W_{j,m}(\zeta) \qquad \text{valid when } -\pi\leq\arg\zeta\leq\pi$$
$$Z_2(\zeta)=W_{-j,m}(\zeta e^{i\pi}) \qquad \text{valid when } -2\pi\leq\arg\zeta\leq 0$$

Putting $c=c_r+ic_i$, it is a known fact that $|c_r|<1$ when $c_i\neq 0$. Restricting ourselves to discussing instability waves, we must further have

(2.26)
$$c_i>0.$$

Assuming first $k^2+r_0>r_1$, we see from (2.12) that

(2.27)
$$-\pi<\arg\zeta_1<-\frac{\pi}{2}, \qquad -\frac{\pi}{2}<\arg\zeta_2<0.$$

(2.15) $$|r_1-r_0|\ll 1, \qquad k^2\ll 1.$$

Then the frequency equation (2.14) reduces to

(2.16) $$\left(\frac{-1-c}{1+c}\right)^{2m}=1$$

where in our approximation, we may write

(2.17) $$m=(\tfrac{1}{4}-r_0)^{\frac{1}{2}}$$

The frequency equation encountered here has been discussed previously by one of the authors (1953c). The result of the discussion may be briefly resumed thus:

For

$$r_0\geq -\tfrac{3}{4},$$

no instability waves with exponentially increasing amplitude occur.

For

$$r_0>\tfrac{1}{4}$$

ordinary stability waves occur. For

$$r_0<-\tfrac{3}{4},$$

a finite number of ordinary instability waves will appear.

c. *Large values of* $|\zeta_1|$ *and* $|\zeta_2|$. For large values of $|\zeta_1|$ and $|\zeta_2|$, which implies

(2.18) $$|\gamma^2|=|k^2+r_0-r_1|\gg 1,$$

we may expand our solutions in asymptotic series. With this in mind we introduce the Whittaker functions defined by the asymptotic expansions ($|\zeta|\gg 1$, finite values of $|j|$ and $|m|$)

(2.19)
$$W_{j,m}(\zeta)=\zeta^j e^{-\frac{1}{2}\zeta}f_j(\zeta), \qquad |\arg\zeta|<\frac{3\pi}{2}$$
$$W_{-j,m}(\zeta e^{\pm i\pi})=(\zeta e^{\pm i\pi})^{-j}e^{\frac{1}{2}\zeta}f_j(-\zeta), \qquad |\arg(\zeta e^{\pm i\pi})|<\frac{3\pi}{2}$$

where

(2.20) $$f_j(\zeta)=1+\sum_{p=1}\frac{\{m^2-(j-\frac{1}{2})^2\}\cdots\{m^2-(j-p+\frac{1}{2})^2\}}{p!\zeta^p}.$$

The asymptotic expansions are inconvenient for arguments in intervals in the neighbourhood of the upper and lower limits. In the following we will consider the expansions in a more narrow region than that given above in order to get a valid approximation by retaining only one or a few terms of the series. With this aim we write

$$Z''+\left[\frac{1}{4}-\frac{ic(r_1-r_0)}{(r_1-r_0-k^2)^{\frac{1}{2}}}\frac{1}{\zeta}+\{r_1+c^2(r_1-r_0)\}\frac{1}{\zeta^2}\right]Z=0\,. \tag{2.6}$$

Here

$$r_1=r(1)\,, \qquad r_0=r(0)\,, \tag{2.7}$$

are Richardson's numbers at the rigid planes, $z=\pm 1$ and at $z=0$ respectively. The primes now denote differentiation with respect to ζ.

With the abbreviations

$$j=-\frac{ic(r_1-r_0)}{(r_1-r_0-k^2)^{\frac{1}{2}}}\,, \qquad m^2=\frac{1}{4}-r_0-c^2(r_1-r_0)\,, \tag{2.8}$$

we obtain finally as differential equation for Z

$$Z''+\left[-\frac{1}{4}+\frac{j}{\zeta}+\frac{\frac{1}{4}-m^2}{\zeta^2}\right]Z=0. \tag{2.9}$$

This is WHITTAKER's form (1962) of the differential equation for confluent hypergeometric functions. Its general solution, assuming $2m$ different from an integer, is

$$Z(\zeta)=AM_{j,m}(\zeta)+BM_{j,-m}(\zeta)\,, \tag{2.10}$$

with A and B arbitrary constants and $M_{j,m}(\zeta)$ given by

$$M_{j,m}(\zeta)=\zeta^{\frac{1}{2}+m}e^{-\frac{1}{2}\zeta}\frac{\Gamma(1+2m)}{\Gamma(\frac{1}{2}+m-j)}\sum_{p=0}^{\infty}\frac{\Gamma(p+\frac{1}{2}+m-j)}{p!\,\Gamma(p+1+2m)}\zeta^p\,. \tag{2.11}$$

In order to obey the kinematic boundary conditions Z must vanish at the rigid planes $z=-1$ and $z=1$, or for

$$\zeta_1=2(k^2+r_0-r_1)^{\frac{1}{2}}(-1-c)\,, \quad \zeta_2=2(k^2+r_0-r_1)^{\frac{1}{2}}(1-c)\,. \tag{2.12}$$

Thus

$$Z(\zeta_1)=Z(\zeta_2)=0\,. \tag{2.13}$$

These equations together with equation (2.10) then give the general frequency equation

$$M_{j,m}(\zeta_1)M_{j,-m}(\zeta_2)-M_{j,-m}(\zeta_1)M_{j,m}(\zeta_2)=0\,. \tag{2.14}$$

In the following we will limit our discussion to the cases that 1) $|\zeta_1|$ and $|\zeta_2|$ are small, and 2) $|\zeta_1|$ and $|\zeta_2|$ are large.

b. *Small values of* $|\zeta_1|$ *and* $|\zeta_2|$. From our expression (2.11) for $M_{j,m}$ we see that for sufficiently small values of $|\zeta|$ and for values of $|j|$ of order 1, we may retain only the first term of the right hand side of (2.11). These two conditions imply that we must have with n given integer values satisfying the relation

$$(n+\tfrac{1}{2})\pi \leq \kappa h. \tag{1.28}$$

According to TOLMIEN (1935) the existence of a neutral wave in a shear flow of an inviscid homogeneous-incompressible fluid implies that exponentially instability waves will occur for wave numbers slightly less than that corresponding to the neutral wave. Previous work by one of the authors (1953b) indicates moreover that for the harmonic velocity profile exponentially instability waves occur for all k-values smaller than that corresponding to the largest k-value giving a neutral wave, excepting the discrete set of k-values corresponding to neutral waves. The discussion is carried through for $k = 0$ and for varying height of the layer.

It seems improbable that a sufficiently small gravitational stability, i.e. a sufficiently small value of a in equation (1.20), should cause a disappearance of the instability occurring for homogeneous fluid ($a = 0$). It may also be reasonably assumed that a similar instability will also occur for the case considered in our first example when neutral waves occur. In the next section we will show this last assumption to be true.

2. The stability of Couetteflow of a fluid with a variation of static stability given by a second degree function of height

a. *The general solution.* Putting in equation (1.6)

$$\beta = az^2 + b, \qquad U = \alpha z, \tag{2.1}$$

we obtain

$$Z'' - \left(k^2 - g\frac{az^2+b}{(\alpha z - c)^2}\right)Z = 0. \tag{2.2}$$

Introducing here the Richardson number r given by

$$r(z) = \frac{g\beta}{\alpha^2} = \frac{g(az^2+b)}{\alpha^2}, \tag{2.3}$$

our equation takes the form

$$Z'' - \left(k^2 - \frac{r(z)}{\left(z - \frac{c}{\alpha}\right)^2}\right)Z = 0. \tag{2.4}$$

We may conveniently choose h as the unit of length and α^{-1} as the unit of time. Introducing further a new independent variable given by

$$\zeta = 2i(r_1 - r_0 - k^2)^{\frac{1}{2}}(z - c), \tag{2.5}$$

equation (2.4) may be written

For $c = 0$ this equation has the solutions

$$Z_1 = A\cos(\kappa^2 + ag - k^2)^{\frac{1}{2}}z$$

(1.21)

$$Z_2 = B\sin(\kappa^2 + ag - k^2)^{\frac{1}{2}}z\,.$$

Instead of the relation (1.13) we now obtain as a necessary requirement for special solutions satisfying the boundary conditions,

$$ag > -\kappa^2\,. \tag{1.22}$$

Thus with the velocity profile (1.18) special solutions satisfying the boundary conditions are possible also for gravitationally unstable stratification.

When

$$h < \frac{\pi}{2\sqrt{\kappa^2 + ag}}\,, \tag{1.23}$$

we do not have any solutions of the considered type of our boundary-value problem.

When

$$\frac{\pi}{\sqrt{\kappa^2 + ag}} > h \geq \frac{\pi}{2\sqrt{\kappa^2 + ag}} \tag{1.24}$$

we have only one solution (the Z_1-type).

When

$$\frac{3\pi}{2\sqrt{\kappa^2 + ag}} > h \geq \frac{\pi}{\sqrt{\kappa^2 + ag}}\,, \tag{1.25}$$

we get one solution of Z_1-type and one of Z_2-type, and so on.

The wave numbers giving special solutions are in the interval given by

$$0 \leq k^2 < \frac{8}{9}(\kappa^2 + ag)\,. \tag{1.26}$$

For the quantity a equal to zero we have a shear-flow of an incompressible and homogeneous ideal fluid with a harmonic velocity profile. Neutral waves of the special type will exist when h is larger than or equal to half the wave-length of the velocity profile. The corresponding wave numbers are given by

$$k^2 = \kappa^2 - \left(\frac{(n+\frac{1}{2})\pi}{h}\right)^2, \tag{1.27}$$

while the second solution Z_2 satisfies the boundary conditions when

$$h^2 = \frac{m^2\pi^2}{ag-k^2} \qquad m=1,2,\ldots, \tag{1.12}$$

In order to obtain special solutions satisfying the boundary conditions we see that the quantity a must necessarily be positive.

$$a>0. \tag{1.13}$$

Thus we must have gravitationally stable stratification. We further see that h must satisfy the relation

$$h \geq \frac{\pi}{2\sqrt{ag}}, \tag{1.14}$$

if special solutions of our boundary value problem shall be possible. For

$$\frac{\pi}{\sqrt{ag}} > h \geq \frac{\pi}{2\sqrt{ag}}, \tag{1.15}$$

we have only one solution (the Z_1-type).

For

$$\frac{3\pi}{2\sqrt{ag}} > h \geq \frac{\pi}{\sqrt{ag}}, \tag{1.16}$$

we have one solution of the Z_1-type and one of the Z_2-type, and so on. The wave numbers giving these solutions must be in the interval given by

$$0 \leq k^2 < \frac{8}{9} ag. \tag{1.17}$$

As another example put

$$U = \alpha \sin \kappa z, \tag{1.18}$$

which implies

$$U'' = -\kappa^2 U. \tag{1.19}$$

Keeping the relation (1.7) unchanged, our equation (1.6) becomes

$$Z'' - \left(k^2 - \kappa^2 \frac{U''}{U-c} - \frac{agU^2}{(U-c)^2}\right) Z = 0. \tag{1.20}$$

Disregarding the kinematic effect of density variations (Boussinesque's approximation) the equation governing two-dimensional perturbation (in the xz-planes) of the basic flow is easily shown to be given by

(1.4)
$$\left(\frac{\partial}{\partial t}+U\frac{\partial}{\partial x}\right)^2\nabla^2\psi-U''\left(\frac{\partial}{\partial t}+U\frac{\partial}{\partial x}\right)\frac{\partial\psi}{\partial x}-\beta(z)g\frac{\partial^2\psi}{\partial x^2}=0.$$

Here ψ is the stream function of the velocity disturbances, $U''=\frac{d^2U}{dz^2}$ and g is the acceleration of gravity.

Attempting a solution of the form

(1.5)
$$\psi=Z(z)e^{ik(x-ct)}$$

we obtain for Z the equation

(1.6)
$$Z''-\left(k^2+\frac{U''}{U-c}-\frac{\beta g}{(U-c)^2}\right)Z=0.$$

If this equation has real Eigen-values c such that U at some level $z=z_0$ in the fluid equals c, the corresponding Eigen solution will in general have a singularity for $z=z_0$. In special cases, however, this singularity will not appear. We then obtain special solutions without singularities and with real values for c. For instance, if we put

(1.7)
$$\beta=aU^2,$$

and

(1.8)
$$U=\alpha z,$$

i.e. Couetteflow, our equation (1.6) takes the form

(1.9)
$$Z''-\left(k^2-\frac{agU^2}{(U-c)^2}\right)Z=0.$$

For $c=0$, this equation has special solutions which may be written

(1.10)
$$Z_1=A\cos(ag-k^2)^{\frac{1}{2}}z,$$
$$Z_2=B\sin(ag-k^2)^{\frac{1}{2}}z.$$

The first solution Z_1 satisfies the boundary conditions when

(1.11)
$$h^2=\frac{(n+\frac{1}{2})^2\pi^2}{ag-k^2},\qquad n=0,1,2,\ldots,$$

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ON THE STABILITY OF SHEAR FLOW OF A STRATIFIED FLUID

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Summary. The equation governing perturbations of a linear flow of a stratified fluid is shown to have 'special' solutions without singularities in special cases of density distribution and velocity profiles. For such cases a stability criterion is suggested (see also HØILAND 1953a) in analogy with the criterion valid for linear flow of a homogeneous fluid when 'special' solutions occur.

The perturbation equation for a special model (linear Couetteflow and a second degree expression with height of the static stability) is solved. By use of asymptotic expansion a frequency equation is developed and discussed. The stability criterion suggested is confirmed for the model. It is also shown that when no special solutions exist, instability waves may occur.

Introduction. In two earlier reports (HØILAND 1953 a, 1954) one of the authors of the present paper took up an investigation of the effect of a continuous variation of gravitational stability on the stability of a linear flow of a stratified fluid. Sections 1, and 2 a and b of the present paper present essentially a review of the results of these investigations. In section 2c a correction is given of the frequency formula in HØILAND (1954) for asymptotic solutions of our governing equation. The asymptotically correct frequency formula is discussed in some detail.

1. Some special solutions of the perturbation equation

Let the basic motion be given by

$$U = U(z), \tag{1.1}$$

in a fluid contained between rigid horizontal planes at the levels

$$z = -h, \qquad z = h. \tag{1.2}$$

Let further the density in the undisturbed state be given by

$$\rho = \rho_0 e^{-\int_0^z \beta(z)dz}, \tag{1.3}$$

and the fluid be considered incompressible.

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