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JØRGEN HOLMBOE

On the instability of stratified shear flow

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ON THE INSTABILITY OF STRATIFIED SHEAR FLOW*

By Jørgen Holmboe

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Abstract. Two simple classical models of shear flow are re-examined in this note, namely (*i*) a layer with constant shear between unbounded irrotational layers and having equal density jumps across the boundaries, and (*ii*) a symmetric jet consisting of two adjacent layers with opposite shear of the same strength between unbounded irrotational layers, having equal density jumps across the boundaries and a different density jump across the center. The main characteristics of the instability of the flow in these models, including the spectral range of unstable waves and the structure and rate of growth of a typical wave in the region, are derived by elementary semi-intuitive considerations without resort to the more laborious conventional analysis of the normal modes of the models.

1. The homogeneous shear layer. By way of introduction to the main topic we consider Lord Rayleigh's classical homogeneous model with the same density in the layers. The unbounded outer irrotational layers have the relative translation $2U$, and the shear layer between them has the depth d and hence the constant vorticity $q=2U/d$.

Let the boundaries of the shear layer be given equal small amplitude sinusoidal deformations with the ordinates $z_s=A\sin kx$ and with the upper wave a quarter wave length downwind from the lower wave. We shall call this wave structure the *c*-state of the wave. From Stoke's theorem the vertical velocity at the nodes of these deformations has the magnitude

$$w=qA\int_{-\infty}^{+\infty}\frac{\sin kxdx}{2\pi x}=\frac{U}{d}A.$$

Both waves move upwind relative to the outer fluid with the same intrinsic speed C which is found from the streamline slope at the nodes in a frame which moves with the wave,

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$$w/C = kA = (U/Cd)A.$$

The waves move intrinsically upwind in the c-state with the speed $C = U/kd$. In particular the wave with the wave length $L = 2\pi d\ (kd = 1)$ is stationary relative to the fluid in the middle of the shear layer. However, the component field from the lower wave makes the amplitude of the upper deformation grow at the rate

$$\dot{A} = e^{-kd} w = (U/d)e^{-1} A; \qquad (kd = 1) \tag{1.1}$$

and the upper field makes the lower amplitude grow at the same rate. Thus, the symmetric $(kd = 1)$-wave is stationary relative to the fluid at the center of the shear layer (the symmetric frame) when the upper deformation is a quarter wave length downwind from the lower deformation. In this stationary c-state the wave amplitude grows at the exponential rate $n = U/de$. With $U = 5\ ms^{-1}$, and $d = 600$ m, the amplitude doubles every 3.75 min.

Let next the symmetric wave have equal deformations in opposite phase. We call this non-tilting state the b-state of the wave. Here the upper and lower fields have the same phase, so the vertical velocity amplitude at the interface nodes is

$$w = \frac{U}{d}(1 + e^{-kd})A = AkC_b.$$

Both the upper and the lower waves move intrinsically upwind with the speed $C_b = (U/kd)(1 + e^{-kd})$. In particular, the wave whose wave number k is given by

$$kd = 1 + e^{-kd} = 1.2785.. \qquad (L = 4.9d) \tag{1.2}$$

is stationary relative to the fluid in the middle of the shear layer. The wave amplitudes do not grow, so this wave has a stationary neutral b-state. Longer waves move in the symmetric frame upwind through the b-state and start growing, shorter waves move downwind and start decaying. The wave in (1.2) with the stationary neutral b-state therefore marks the boundary between the longer unstable and the shorter stable waves.

The $(kd = 1)$-wave in (1.1) with the stationary growing c-state has the most favorable tilt for effective growth. In the spectral interval between the stability boundary and $kd = 1$ the waves have stationary tilts between the b-state and the c-state, so they grow at a slower rate than the $(kd = 1)$-wave, both because their tilts are less favorable for growth and because the exponential reduction of the fields between the boundaries is greater. The longer $(kd < 1)$-waves have tilts between the c-state and the *a-state* (deformations in the same phase), again less favorable for growth. But here the field reduction between the boundaries is smaller than in the $(kd = 1)$-wave. The fastest growing wave is therefore somewhat longer than $kd = 1$.

These qualitative results are confirmed by Lord Rayleigh's linearized theory. The wave number and growth rate of the fastest growing wave* are given by

$$kd=1-e^{-2kd}=0.8;\quad n=\dot{A}/A=0.4U/d.$$

It is 20 per cent longer and grows 9 per cent faster than the ($kd=1$)-wave. It tilts 63° 30′ downwind from the lower to the upper crest (wave length =360°).

The spectral position of the stability boundary (1.2) and the position and growth of the wave with the stationary c-state (1.1) were obtained with little labor. These items alone give a fairly good idea of the stability properties of the shear layer.

2. The homogeneous four-layer jet. There is no motion in the two unbounded outer layers. Between them is a symmetric two-layer jet of finite depth d with the fluid in the center of the jet moving with the speed U relative to the fluid in the outer layers, so the two layers of the jet have the same constant shear (vorticity) $2U/d$ in opposite directions above and below the center level. All four layers have the same density.

Let the boundaries of the jet have small-amplitude sinusoidal deformations with the same phase and the same amplitude A_s, and the center interface a sinusoidal deformation with the same wave length a quarter wave length downwind from the boundary deformations and the amplitude A_0. As in the shear layer, we call this the c-state of the wave. The vertical velocity amplitudes at the nodes of the deformations are

Boundaries: $$w_s=\frac{U}{d}(1+e^{-kd})A_s=C_skA_s,$$

Center: $$w_0=\frac{2U}{d}A_0 \qquad =C_0kA_0.$$

The boundary deformations propagate intrinsically downwind relative to the outer fluid with the speed C_s. The center deformation propagates intrinsically upwind with the speed C_0. These are determined by the slopes of the deformations at the nodes, as indicated. Let us choose the wave length for which the sum of these speeds is the speed U in the center of the jet so the center and boundary waves have no relative motion. With the use of the abbreviations

(2.1) $$\kappa=kd;\quad \alpha=e^{-kd}=e^{-\kappa},$$

the wave number of the wave with this stationary c-state is given by

(2.2) $$C_s+C_0=U=U(3+\alpha)/\kappa;\quad \kappa=3+\alpha=3.05.$$

The wave is stationary in a frame which moves relative to the outer fluid with the

* See Appendix A (4).

speed C, which may be written

$$C=\tfrac{1}{2}[C_s+(U-C_0)]=\tfrac{1}{2}\left(1-\frac{1-\alpha}{\kappa}\right)U=\frac{1+\alpha}{3+\alpha}U\sim\frac{1}{3}U. \tag{2.3}$$

We call this frame *the characteristic frame* of the wave. It plays the same role in the jet as the symmetric frame in the shear layer.

The wave is stationary for arbitrary values of the deformation amplitudes. However, only when the amplitudes are the same, $A_s=A_0=A$, will the interactions between the center and the boundaries be the same and make the amplitudes grow at the same rate, $\dot{A}=2\sqrt{\alpha}\,(U/d)A$, so the wave as a whole grows at the exponential rate

$$n=\dot{A}/A=2\sqrt{\alpha}(U/d)\approx 2/\sqrt{e}(U/ed). \qquad (\kappa=3+\alpha) \tag{2.4}$$

It is about 20 per cent faster than the ($\kappa=1$)-wave in a shear layer with the same vorticity (see 1.1). The jet is more unstable than the shear layer.

In the *b*-state of the wave the center deformation has opposite phase to the boundary deformations and the fields have the same phase. The vertical velocity amplitudes at the nodes are then

$$w_s=\frac{U}{d}(1+\alpha+2\sqrt{\alpha})A=C_skA,$$

$$w_0=\frac{U}{d}(2+2\sqrt{\alpha})A=C_0kA.$$

The deformations have the same amplitude to make the interactions between the center and the boundaries equal. The wave number of the wave which has a stationary *b*-state, as in (2.2), is given by

$$\kappa=3+\alpha+4\sqrt{\alpha}=3.67 \qquad \text{(stability boundary)} \tag{2.5}$$

The wave is stationary in a frame which moves relative to the outer fluid with the speed

$$C=\tfrac{1}{2}[C_s+(U-C_0)]=\tfrac{1}{2}\left(1-\frac{1-\alpha}{\kappa}\right)U=\frac{1+\alpha+2\sqrt{\alpha}}{3+\alpha+4\sqrt{\alpha}}U. \tag{2.6}$$

The characteristic frame is related to the wave length by the same formula as for the wave with the stationary *c*-state in (2.3). Using the same formula for all wave lengths it is readily seen that waves longer than the stationary wave in (2.5) move symmetrically upwind in the characteristic frame through the *b*-state and start growing, shorter waves move symmetrically downwind and start decaying. The wave in (2.5) with the stationary *b*-state is therefore the spectral boundary between the shorter stable and the longer unstable waves.

By the same considerations as for the shear layer it is evident that the fastest growing wave in the jet is somewhat longer and grows a little faster than the wave with the stationary c-state. The precise values are obtained from the linear wave theory in Appendix B. They are

wave number: $\kappa=3+\alpha-8\alpha/(1+\alpha)=2.45,$

max. growth rate: $n=2\sqrt{\alpha}(1-\alpha)/(1+\alpha)\frac{U}{d}=0.495\frac{U}{d},$

tilt: $\cos 2\sigma_s=2\sqrt{\alpha}/(1+\alpha);\quad 2\sigma_s=57^0.$

Again we note that the position of the stability boundary (2.5) and the position and growth of the wave which has a stationary c-state (2.4) give a fairly good idea of the stability properties of the jet.

3. The stratified shear layer. Let now the fluid above and below the shear layer in section 1 have different densities. The unbounded layer below the shear layer has the constant density ρ_2 and the unbounded layer above the shear layer has the smaller density ρ_1, while the density in the shear layer is the arithmetic mean of these densities. It will be assumed that the density difference across the boundaries is a small fraction of the mean density, so the kinematic effect of the density difference may be ignored.

The static stability of the shear layer is measured by the non-dimensional parameter

$$s=(\rho_2-\rho_1)/(\rho_2+\rho_1)\ll 1. \tag{3.1}$$

The Richardson number of the shear layer, that is, the non-dimensional ratio between static stability and shear, is defined by the formula,

$$\mu=\tfrac{1}{2}sgd/U^2. \tag{3.2}$$

When this shear layer is disturbed, the "overweight" associated with the slopes of the boundaries gives rise to sliding vorticity changes along the boundaries. Let at a certain time the upper boundary have the sinusoidal deformation

$$z_s=A\sin kx, \qquad \text{(upper deformation)}$$

and along the deformed boundary the sliding vorticity distribution

$$u^+-u^-=(2U/d)A_g\sin(kx+\beta).$$

For short we call this sliding vorticity field a gravity wave (see Fig. 3). Its intrinsic

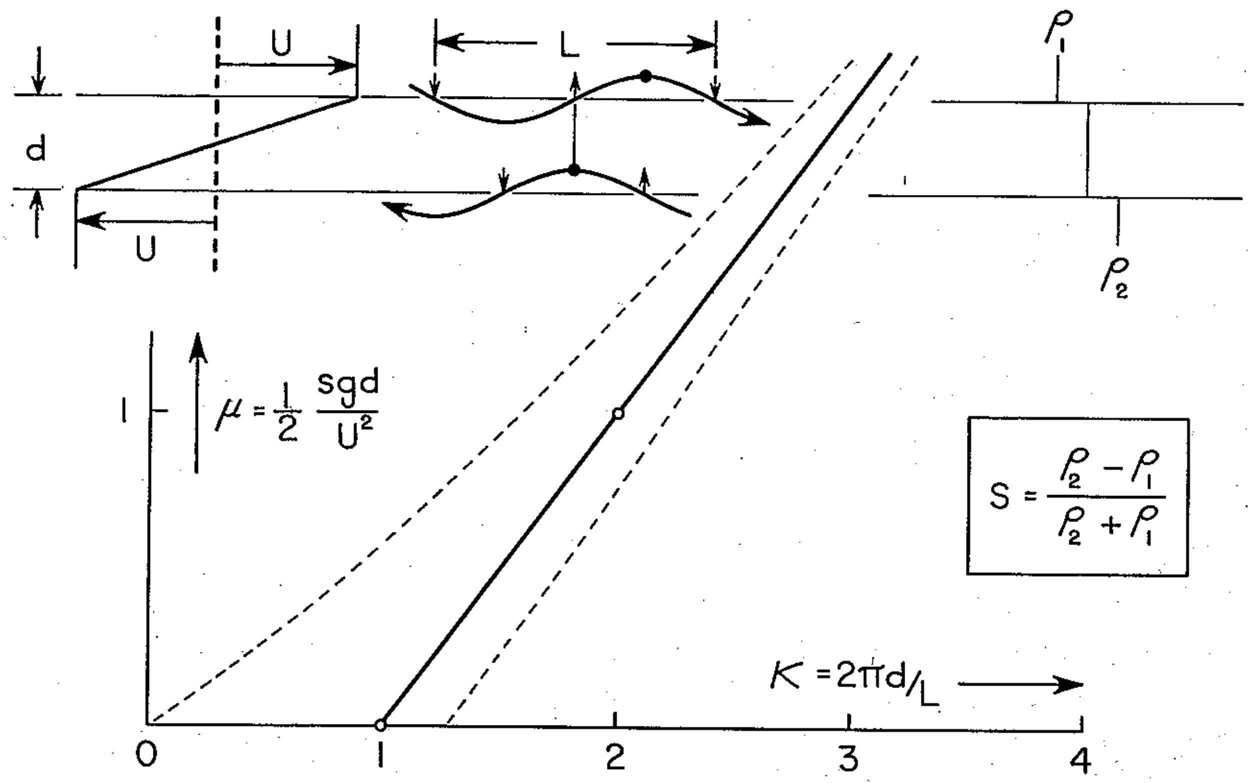

Fig. 1. Stability diagram for stratified shear layer.

local change is governed by the slope of the statically stable boundary in accordance with the dynamic condition,

$$\frac{\partial}{\partial t}(u^+ - u^-) = sg\frac{\partial z_s}{\partial x},$$

which, with the above values substituted, gives

$$\frac{d}{dt}(A_g \cos\beta) = 0 \; ; \qquad \frac{d}{dt}(A_g \sin\beta) = kU\mu A,$$

or more explicitly

(3.3) $$\dot{A}_g / A_g = \dot{\beta} \tan\beta$$

(3.4) $$\dot{\beta} A_g = kU\mu A \cos\beta.$$

We note that the growth and propagation of the gravity wave is determined locally by the amplitude and phase of the boundary deformation, so these equations apply to the gravity waves at both boundaries. In particular, the gravity wave is stationary in the symmetric frame ($\dot{\beta} = kU$) when its amplitude has the value $A_g = \mu A \cos\beta$.

Consider now a simple *c*-state of a symmetric wave with the upper deformation a quarter wave length downwind from the lower deformation, and the gravity waves in phase with the shear waves ($\beta=0$) so the gravity waves are instantaneously neutral. To make this wave stationary in the symmetric frame both the gravity wave and the shear wave (the boundary deformations) must be stationary. The gravity wave is stationary when its amplitude has the value $A_g=\mu A$, so the vertical velocity at the nodes has the value $w=(U/d)(1+\mu)A$. The shear wave is stationary when the wave has the wave length which makes

(3.5) $$w=(U/d)(1+\mu)A=UkA\,; \qquad \kappa=1+\mu\,.$$

However the wave will not remain stationary in this state, for the deformation amplitudes grow at the rate

(3.6) $$\dot{A}/A=\alpha(1+\mu)(U/d)$$

while the gravity wave is neutral. In the next instant $A_g<\mu A$, and therefore from (3.4) $\dot{\beta}>Uk$, so the gravity wave begins to move *upwind* from the initial c-state. At the same time the intrinsic speed of the shear wave is reduced (see 3.5), so the shear wave begins to move *downwind* from the initial c-state. As the phase difference β between gravity wave and shear wave develops from these displacements the gravity wave begins to grow at an increasing rate and its upwind speed decreases again, while the initial growth rate of the shear wave is reduced by the opposing action of the local gravity wave. Through these adjustments the $(\kappa=1+\mu)$-wave approaches a stationary state with all wave amplitudes growing at the same constant exponential rate n which is less than the initial growth rate of the shear wave in (3.6),

(3.7) $$n=\dot{A}/A=\dot{A}_g/A_g=Uk\tan\beta<Uk\alpha\,.$$

The gravity wave has now the amplitude $A_g=\mu A\cos\beta$. With γ denoting the downwind phase of the shear wave from the initial c-state (see Fig. 3), the growth rate of the shear wave is $n=\dot{A}/A=\alpha(U/d)[\cos\gamma+\mu\cos\beta\cos(\gamma-\beta)]-(U/d)\mu\cos\beta\sin\beta$. From these we get one relation between β and γ,

(3.8) $$(1+\mu)\tan\beta+\mu\sin\beta\cos\beta=\alpha[\cos\gamma+\mu\cos\beta\cos(\gamma-\beta)]\,.$$

A second relation comes from the kinematic condition at the node,

$$w=UkA=(U/d)(1+\mu\cos^2\beta)A+\alpha(U/d)[\sin\gamma+\mu\cos\beta\sin(\gamma-\beta)]A$$

which gives

(3.9) $$\mu\sin^2\beta=\alpha[\sin\gamma+\mu\cos\beta\sin(\gamma-\beta)]\,.$$

The three equations (3.7–9) determine the growth rate in the stationary growing state (the normal mode) of the $(\kappa=1+\mu)$-wave, and the phase adjustments from the c-state to the mode. The solution is very simple if the shear layer has the static stability for which $\gamma=\beta$, such that the gravity wave and the shear wave have equal displacements upwind and downwind from the c-state. The wave number, phase, and growth rate of this mode are

$$\kappa=1+\mu=\mu^2-\alpha^2=2.6204\,..$$

(3.10)
$$\beta=\gamma=\tan^{-1}(\alpha/\sqrt{\kappa})=2^{\circ}35'$$

$$n=\alpha\sqrt{\kappa}\,(U/d)=0.1178(U/d)\,.$$

Since from (3.7) $\beta<\alpha$, we may for large μ (strong static stability) develop the sines and consines in series and ignore higher powers of the phase angles. Let the subscript o denote the corresponding approximate values. The approximate phase adjustments from the c-state are

(3.11)
$$\frac{\beta_o}{\alpha}=\frac{1+\mu}{1+2\mu}\,;\qquad \frac{\gamma_o}{\beta_o}=\frac{\mu}{1+\mu}\,\frac{2+3\mu}{1+2\mu}$$

and the approximate growth rate is

(3.12)
$$n_o=\alpha\frac{(1+\mu)^2}{1+2\mu}\,\frac{U}{d}=\frac{\alpha\kappa^2}{2\kappa-1}\,\frac{U}{d}\,.$$

We shall examine the accuracy of these approximate values by comparing them with the correct values which are derived below by a different method.

Let us first find the spectral boundaries between the unstable and the stable waves. The waves on these boundaries are neutral and stationary in the symmetric frame. The gravity wave is neutral when it is in phase with the shear wave ($\beta=0$, see 3.3), and it is stationary in the symmetric frame when its amplitude is $A_g=\mu A$ (see 3.4). The shear wave is neutral when it has no tilt with the deformations of the boundaries in phase (a-state) or in opposite phase (b-state). The vertical velocities at the nodes in these two neutral non-tilting states are

$$w=(U/d)(1+\mu)(1\pm\alpha)A=CkA\,,$$

where C denotes the intrinsic upwind speed of the shear wave. The two neutral waves which are stationary in the symmetric frame ($C=U$) have the wave numbers:

(3.13)
Stationary a-state: $n_a(1+\mu)=1\,,$ $\qquad n_a=(1-\alpha)/\kappa$

Stationary b-state: $n_b(1+\mu)=1\,.$ $\qquad n_b=(1+\alpha)/\kappa$

The corresponding lines in a κ,μ-diagram are shown by the dashed lines in Fig. 1. They are the boundaries of the unstable region. Both lines have the same asymptote, $\kappa=\mu+1$. The a-line starts from the origin with the slope $\frac{1}{2}$. The b-line starts from $\kappa=1.2785\ldots$ with unit slope.

Since there are two stability boundaries, the general growth rate equation must be bi-quadratic with a double root $n^2=0$ on each stability boundary. In other words it must have the form

$$[n^2-1+n_a(1+\mu)][n^2-1+n_b(1+\mu)]+Bn^2=0; \qquad (Uk=1)$$

where n is the non-dimensional growth rate measured in units of Uk. The value of B is found from the two limiting systems:

$$U\to 0: \quad -(n^2/\mu+n_a)(n^2/\mu+n_b)=(n^2/\mu)(B/\mu)=0.$$

$$s\to 0: \quad n^4+n^2(B+n_a+n_b-2)+(1-n_a)(1-n_b)=0.$$

The first of these gives the frequencies of the two families of gravity waves in the resting three layer system, namely

$$\left(\frac{m}{kU}\right)^2=-\left(\frac{n}{kU}\right)^2=\mu n_{a,b}=\frac{\frac{1}{2}sg}{k}\frac{1\pm\alpha}{U^2}; \qquad (U\to 0)$$

and shows that B is independent of μ. In the statically neutral shear layer (see Appendix A) one root of the growth rate equation is $n^2=-(1-n_a)(1-n_b)$, so the other root is here $n^2=-1$. The value of B is therefore $(2-n_a)(2-n_b)$, and the growth rate equation for the statically stable shear layer is

$$\text{(3.14)} \quad \left[\left(\frac{n}{kU}\right)^2-1+(1-\alpha)\frac{1+\mu}{\kappa}\right]\left[\left(\frac{n}{kU}\right)^2-1+(1+\alpha)\frac{1+\mu}{\kappa}\right]+\left(\frac{n}{kU}\right)^2\left(2-\frac{1-\alpha}{\kappa}\right)\left(2-\frac{1+\alpha}{\kappa}\right)=0.$$

This is Goldstein's growth rate equation for the shear layer which he derived in 1931. For waves on the $(\kappa=1+\mu)$-line it has the simple form

$$\text{(3.15)} \qquad \left(\frac{nd}{U}\right)^4+\left(\frac{nd}{U}\right)^2[(2\kappa-1)^2-\alpha^2]=\alpha^2\kappa^4. \qquad (\kappa=1+\mu)$$

We note that the $(\kappa=\mu^2-\alpha^2)$-wave has the growth rate $n=\alpha\sqrt{\kappa}\,(U/d)$, as derived earlier in (3.10). For other $(\kappa=1+\mu)$-waves the growth rates are shown by the solid line in Fig. 2. The percentage error ε of the approximate value n_0 in (3.12), is shown by the dashed line. The approximation overestimates the growth rate a little, but the error is less than 0.2 per cent for all $(\kappa=1+\mu)$-waves.

With the value of n known the value of the phase difference β between the gravity wave and the shear wave is obtained from (3.7) and shown in Fig. 3. The corresponding percentage error $\varepsilon(\beta)$ of the value in (3.11) is shown by the dashed line. Finally with β known the phase adjustment γ of the shear wave from the c-state is obtained from (3.8) or (3.9). Using both we get

$$\gamma=\tan^{-1}\left(\frac{\mu\sin\beta\cos\beta}{1+\mu\cos^2\beta}\right)+\tan^{-1}\left(\frac{\mu\sin\beta\cos\beta}{1+\mu+\mu\cos^2\beta}\right),$$

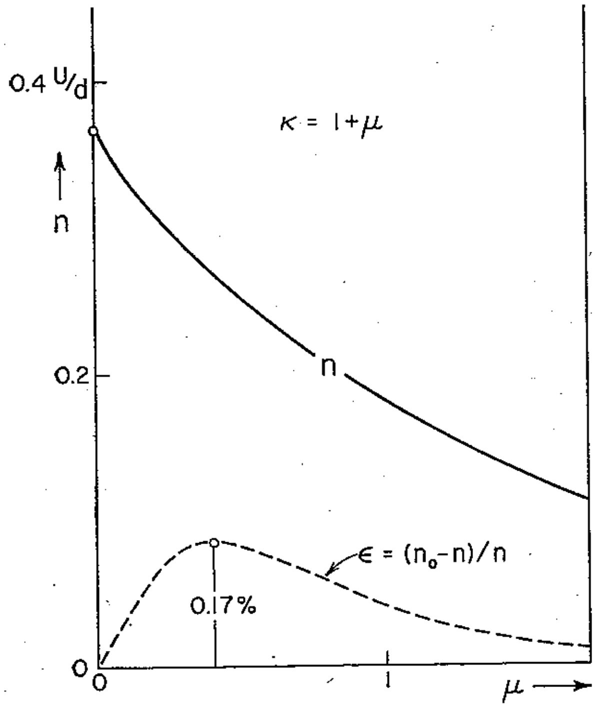

Fig. 2. Growth rate for waves on c-line.

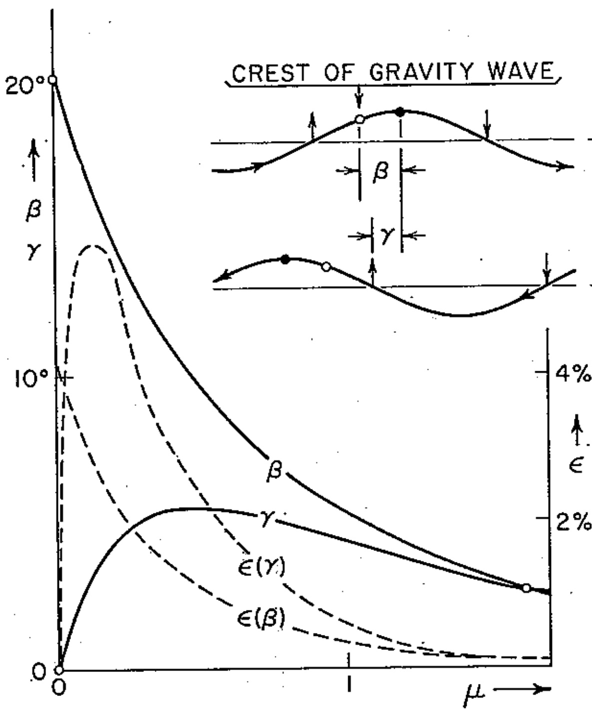

Fig. 3. Phase adjustments from c-state to growing mode.

which reduces to the value in (3.11) when higher powers of β are ignored. The values predicted from the linear approximations of the phase angles are surprisingly good for all $(\kappa = 1 + \mu)$-waves.

4. The stratified four-layer jet. The layers have different constant densities decreasing upward from layer to layer. The density difference between the outer layers is divided between equal density jumps $\Delta\rho_s$ across the boundaries and the remainder $\Delta\rho_0$ across the center of the jet,

$$\rho_2 - \rho_1 = 2\Delta\rho_s + \Delta\rho_0 \ll \rho_2 + \rho_1 .$$

The ratios of these double mean densities are the corresponding non-dimensional static stability parameters

$$s = 2\sigma_s + \sigma_0 \ll 1 . \tag{4.1}$$

The dynamic properties of the stratified jet are determined by two non-dimensional parameters, the Richardson number of the jet and the ratio between the static stabilities at the center and the boundaries of the jet,

$$\mu = \tfrac{1}{2} sgd/U^2 ; \qquad r = 2\sigma_s/s . \tag{4.2}$$

With the same notations as in the shear layer (see 3.3–4) the local intrinsic changes of the amplitude and phase of the gravity waves associated with the static stabilities are

$$\dot{A}_g/A_g = \dot{\beta}\tan\beta, \tag{4.3}$$

$$\dot{\beta}A_g = (k\sigma g d/U)A\cos\beta. \tag{4.4}$$

Consider first a wave in a simple c-state with the center deformation a quarter wave length downwind from the boundary deformations, and the gravity wave in phase with the shear wave ($\beta=0$). Let us choose the wave length for which this wave is stationary in a characteristic frame which moves downwind with a certain speed C relative to the outer fluid. The gravity wave in the boundaries is stationary in this frame, $\dot{\beta}_s=kC$, when its amplitude is

$$A_{gs} = \frac{\sigma_s g d}{UC}A_s = \mu U A_s \frac{r}{C}.$$

The gravity wave in the center is stationary, $\dot{\beta}_0 = k(U-C)$, when its amplitude is

$$A_{g0} = \frac{\sigma_0 g d}{U(U-C)}A_0 = 2\mu U A_0 \frac{1-r}{U-C}.$$

The kinematic conditions at the nodes of the wave with the stationary c-state are therefore

$$\begin{aligned} w_s &= \frac{U}{d}\left[1+\mu U\frac{r}{C}\right](1+\alpha)A_s = kA_sC, \\ w_0 &= \frac{2U}{d}\left[1+\mu U\frac{1-r}{U-C}\right]A_0 = kA_0(U-C). \end{aligned} \tag{4.5}$$

These give two expressions for the speed of the characteristic frame, namely

$$\begin{aligned} 2\kappa C/U &= (1+\alpha)[1+\sqrt{1+4\kappa\mu r/(1+\alpha)}], \\ \kappa(1-C/U) &= 1+\sqrt{1+2\kappa\mu(1-r)}. \end{aligned} \tag{4.6}$$

Elimination of C gives the wave number of the wave with the stationary c-state, namely

$$2\kappa-3-\alpha = (1+\alpha)\sqrt{1+4\kappa\mu r/(1+\alpha)} + 2\sqrt{1+2\kappa\mu(1-r)}. \tag{4.7}$$

The static stability ratio r may have any value between 0 and 1. For every value of r the wave number of the stationary c-state in (4.7) is represented in a κ,μ-diagram by a line which we shall call a c-line (fig. 4). All the c-lines start from the point $\kappa=3+\alpha$

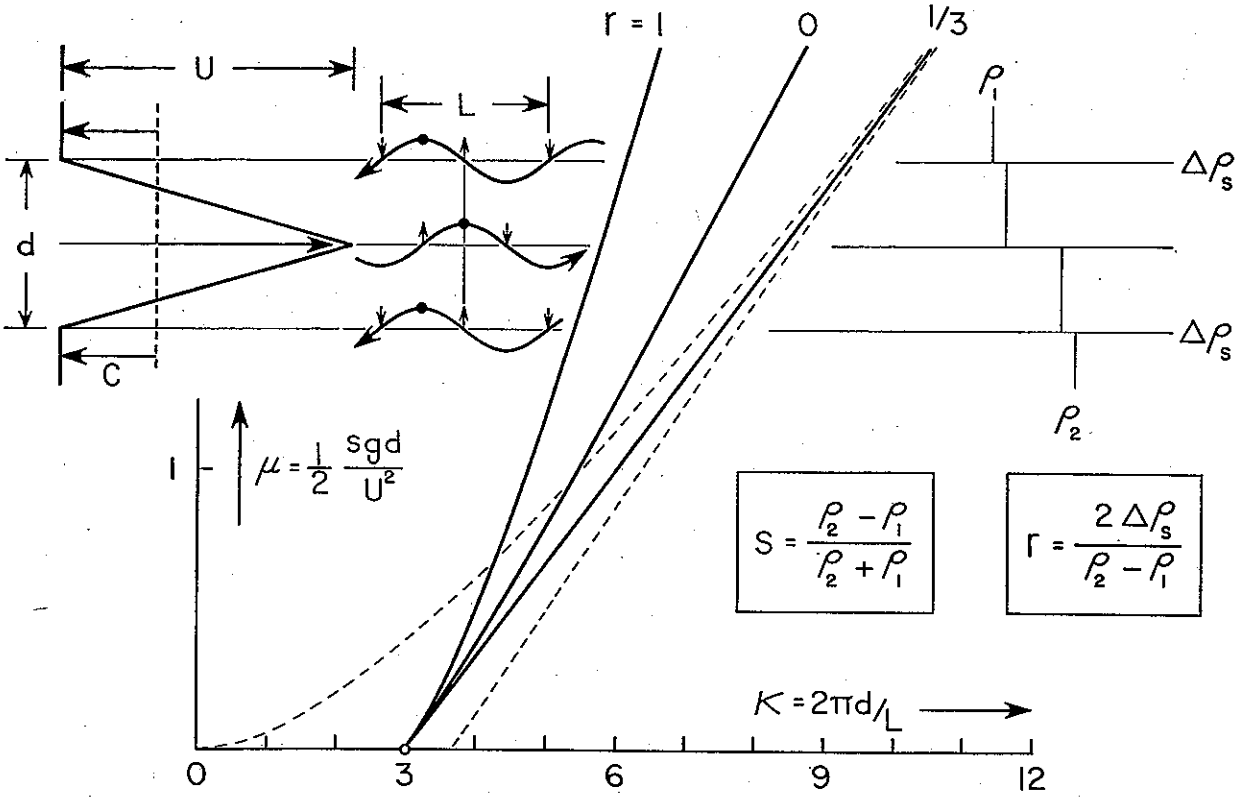

Fig. 4. Stability diagram for stratified jet.

on the κ-axis with the slope $d\mu/d\kappa=(1+\alpha)/\kappa$. For larger κ we may ignore $\alpha=\exp(-\kappa)$ which comes from the interactions between the boundaries. The c-line is very nearly coincident with the algebraic curve

$$2\kappa-3=\sqrt{1+4\kappa\mu r}+2\sqrt{1+2\kappa\mu(1-r)},$$

which has an asymptote with the slope $\kappa/\mu=2-r+\sqrt{8r(1-r)}$. The asymptote has the maximum slope $\kappa/\mu=3$ for $r=\frac{1}{3}$.

In the jets with the limiting values of the static stability ratio the equation (4.7) for the c-line has simpler forms,

The $r=1$ c-line: $\quad(\kappa-2)(\kappa-3-\alpha)=\kappa\mu(1+\alpha).$

The $r=0$ c-line: $\quad(\kappa-1-\alpha)(\kappa-3-\alpha)=2\kappa\mu.$

They are shown in the κ,μ-diagram in Fig. 4. Apart from the slight α-modification near the lower end they are very nearly hyperbolas with the asymptotes $\kappa=\mu+5$ and $\kappa=2\mu+4$.

The third c-line in Fig. 4 represents a jet with the static stability ratio $r=(1+\alpha)/(3+\alpha)$ It has the equation

$r=(1+\alpha)(3+\alpha)$ c-line: $\quad\kappa=(3+\alpha)(1+\mu). \quad (r\approx 1/3).$

If we ignore the slight α-effect this is a straight line with the maximum c-line slope, $\kappa/\mu=3$, and the static stability ratio is very nearly one-third. That is, two-thirds of the density difference between the outer layers is located across the center of the jet, and the remaining third is divided in equal jumps across the boundaries. We shall for short call this the $\left(r=\frac{1}{3}\right)$-jet. It is interesting to note that its c-line lies far outside and to the right of the sector bounded by the c-lines of the $(r=0$ and $1)$-jets.

When the center and boundary deformations in the stationary c-state have the amplitude ratio

$$\left(\frac{A_0}{A_s}\right)^2=\frac{1+\mu Ur/C}{1+\mu U(1-r)/(U-C)}, \tag{4.8}$$

they grow at the same rate

$$\frac{\dot{A}_0}{A_0}=\frac{\dot{A}_s}{A_s}=2\sqrt{\alpha}\frac{U}{d}\sqrt{\left(1+\mu U\frac{r}{C}\right)\left(1+\mu U\frac{1-r}{1-C}\right)}, \tag{4.9}$$

where C has the value in (4.6). The wave quickly adjusts itself from this state toward the asymptotic state of the growing mode with all amplitudes growing at the same constant exponential rate. We shall examine this adjustment in some detail in the $\left(r=\frac{1}{3}\right)$-jet which in a way is quite similar to the corresponding adjustment in the shear layer.

5. The (r=1/3)-jet. The static stability ratio in the jet is in fact slightly larger than one-third, $r=(1+\alpha)/(3+\alpha)$. The jet may be defined as having the static stability ratio $r=C/U$, where C is the speed of the characteristic frame of the stationary c-state in (4.5). The corresponding expressions for C from (4.5) are

$$\begin{aligned} \kappa C/U&=(1+\alpha)(1+\mu), \\ \kappa(1-C/U)&=2(1+\mu). \end{aligned} \tag{5.1}$$

Elimination of C gives the earlier values

c-line: $\kappa=(3+\alpha)(1+\mu)$;
static stability ratio: $r=(1+\alpha)/(3+\alpha)=C/U$.

The center and boundary deformations grow at the same rate in the c-state when they have equal amplitudes $A_0=A_s=A$. The growth rate is

$$\dot{A}/A=2\sqrt{\alpha}\,(1+\mu)(U/d).$$

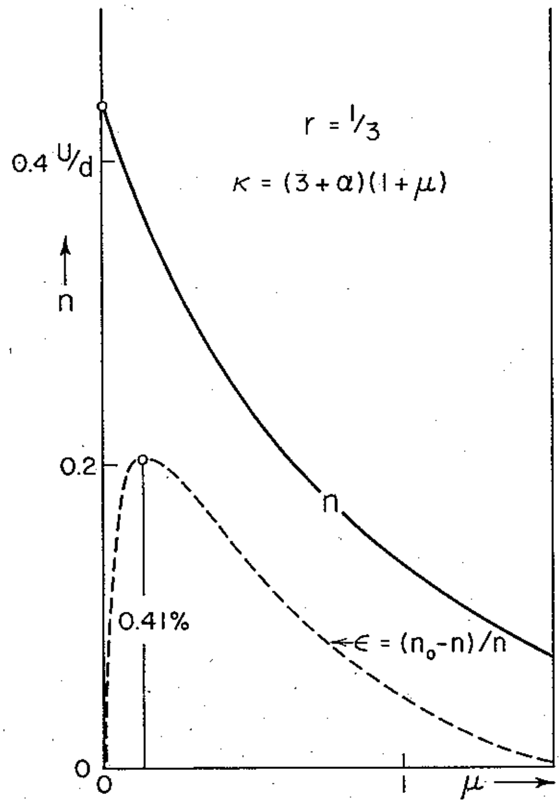

Fig. 5. Growth rate for waves on c-line.

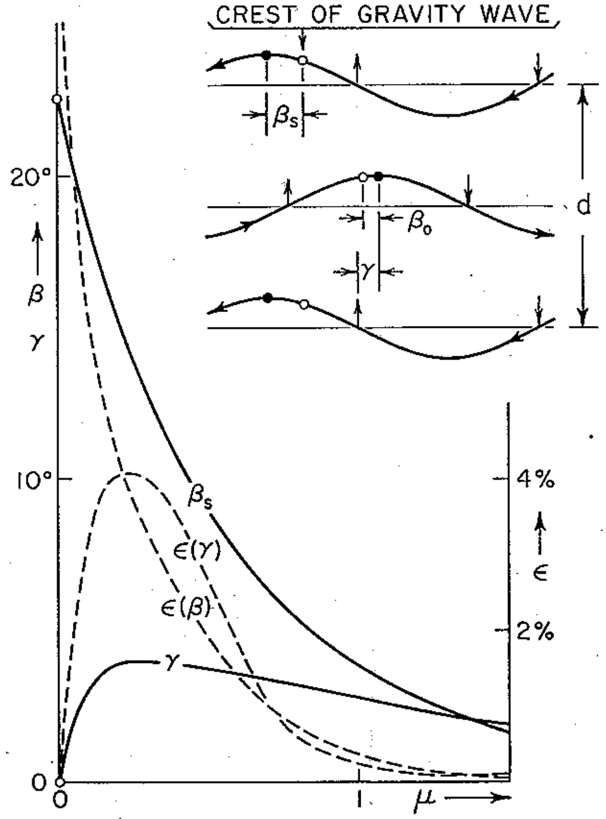

Fig. 6. Phase adjustments from c-state to growing mode.

After the adjustment toward the asymptotic state of the growing mode (see Fig. 6) all amplitudes grow at the same rate

$$n = \dot{A}/A = \dot{A}_g/A_g = kC \tan\beta_s = k(U-C)\tan\beta_0 \,. \tag{5.2}$$

Assume provisionally that the speed of the characteristic frame does not change significantly during the adjustment. The amplitudes of the gravity waves are then $A_g = A\mu \cos\beta$. The kinematic conditions at the crests of the center and boundary deformations, using (5.2), are (see 3.8).

$$2[(1+\mu)\tan\beta_0 + \mu\cos\beta_0 \sin\beta_0] = 2\sqrt{\alpha}[\cos\gamma + \mu\cos\beta_s \cos(\gamma-\beta_s)]A_s/A_0 \,,$$

$$(1+\alpha)[(1+\mu)\tan\beta_s + \mu\cos\beta_s \sin\beta_s] = 2\sqrt{\alpha}[\cos\gamma + \mu\cos\beta_0 \cos(\gamma-\beta_0)]A_0/A_s \,.$$

To the linear approximation of the phase angles these give

$$A_s = A_0 \,, \quad \text{and} \quad \beta_s = 2\beta_0 = 2\sqrt{\alpha}(1+\mu)/(1+2\mu) \tag{5.3}$$

and hence the approximate growth rate

$$n_0 = 2\sqrt{\alpha}\frac{(1+\mu)^2}{1+2\mu}\cdot\frac{U}{d}, \tag{5.4}$$

precisely as in the shear layer (see 3.12).

The right value of the growth rate for the $\kappa=(3+\alpha)(1+\mu)$-mode in a jet with the static stability ratio $r=(1+\alpha)/(3+\alpha)$, as obtained from the quartic factor of the frequency equation in Appendix C, is shown by the solid line in Fig. 5. The percentage error $\varepsilon=(n_0-n)/n$ of the approximate value in (5.4) is indicated by the dashed line. The error is less than 0.4 per cent for all values of μ.

With the growth rate known the right value of the phase adjustment β_s is obtained from (5.2). It is shown in Fig. 6 by the solid line marked β_s. The percentage error $\varepsilon(\beta)$ of the approximate value in (5.3) is indicated by a dashed line.

Finally, the phase adjustment γ of the shear wave and the change of the speed of the characteristic frame, $C_m=C(1+\delta)$, are obtained from the kinematic conditions at the nodes. These conditions, to the second order of the phase angles, may be written

center: $$(1+2\mu)(\gamma\beta_s+\delta)=\tfrac{1}{2}\mu\frac{3+5\mu}{1+\mu}\beta_s^2,$$

boundaries: $$(1+2\mu)(\gamma\beta_s-\delta)=\tfrac{1}{2}\mu\frac{3+4\mu}{1+\mu}\beta_s^2,$$

which give

$$\frac{\gamma}{\beta_s}=\tfrac{3}{4}\cdot\frac{\mu}{1+\mu}\cdot\frac{2+3\mu}{1+2\mu};\quad \frac{\delta}{\gamma\beta_s}=\frac{\mu}{3(2+3\mu)}. \tag{5.5}$$

The percentage change of the speed of the characteristic frame is an order less than the phase adjustments. The right values of γ, as obtained from the theory in Appendix C, are shown in Fig. 6 by the solid line marked γ, and the percentage error $\varepsilon(\gamma)$ of the value in (5.5) is indicated by the dashed line. The errors in the phase adjustments are of the same order as the corresponding errors in the shear layer. Only for small values of the static stability do they exceed a few per cent.

6. The (r=1)-jet. This jet has no density jump across the center. The density difference between the outer layers is divided in equal jumps across the boundaries of the jet. The kinematic conditions in (4.5) for the wave with the stationary c-state in this jet become

$$\kappa C/U=(1+\mu U/C)(1+\alpha)=\kappa-2. \tag{6.1}$$

The wave number is given by

$$(\kappa-3-\alpha)(\kappa-2)=\kappa\mu(1+\alpha). \qquad (c\text{-line}) \tag{6.2}$$

When the deformation amplitudes have the ratio

$$A_0/A_s = \sqrt{1+\mu U/C},$$

they grow at the same rate, namely

$$\dot{A}_0/A_0 = \dot{A}_s/A_s = 2\sqrt{\alpha}\sqrt{1+\mu U/C}(U/d).$$

After the adjustment from this c-state toward the asymptotic state of the growing mode all wave amplitudes grow at the same rate as the gravity wave in the boundaries,

$$n = \dot{A}_0/A_0 = \dot{A}_s/A_s = \dot{A}_g/A_g = kC\tan\beta. \tag{6.3}$$

where the gravity wave has the amplitude $A_g = (\mu U/C)A_s \cos\beta$.

Assuming again that we may ignore the slight change in the speed of the characteristic frame during the adjustment, the kinematic conditions at the wave crests are

$$(1+\alpha)(1+\mu U\ C)\tan\beta = 2\sqrt{\alpha}[\cos\gamma + (\mu U/C)\cos\beta\cos(\gamma-\beta)]A_s/A_0,$$

$$(1+\alpha)[(1+\mu U/C)\tan\beta + (\mu U/C)\cos\beta\sin\beta] = (2\sqrt{\alpha}\cos\gamma)A_0/A_s.$$

To the linear approximation of the phase angles these give

$$A_0/A_s = \sqrt{1+2\mu U/C}\ ; \qquad \beta = 2\sqrt{\alpha}/\sqrt{1+2\mu U/C}. \tag{6.4}$$

and hence from (6.3) the approximate growth rate

$$n_0 = 2\sqrt{\alpha}\frac{1+\mu U/C}{\sqrt{1+2\mu U/C}}\frac{U}{d}. \tag{6.5}$$

During the adjustment the center deformation grows faster than the boundary deformations, whose growth is opposed by the gravity wave, so the amplitude ratio is larger in the mode than in the initial c-state.

The right value of the growth rate, as obtained from the frequency equation in Appendix C, is shown by the solid line in Fig. 7, and the percentage error ε of the approximate value in (6.5) is indicated by the dashed line.

With the growth rate known the right value of the phase adjustment β is obtained from (6.3). It is shown in Fig. 8 by the solid line marked β, and the percentage error of the value in (6.4) is indicated by the dashed line.

The phase adjustment γ and the change of the speed of the characteristic frame, $C_m = C(1+\delta)$, are obtained from the kinematic conditions at the nodes. To the second order of the phase angles they are

center: $$(1+\mu U/C)(\beta\gamma+\delta) = \beta^2\mu U/C,$$

boundaries: $$(1+2\mu U/C)(\beta\gamma-\delta) = \beta^2\mu U/C.$$

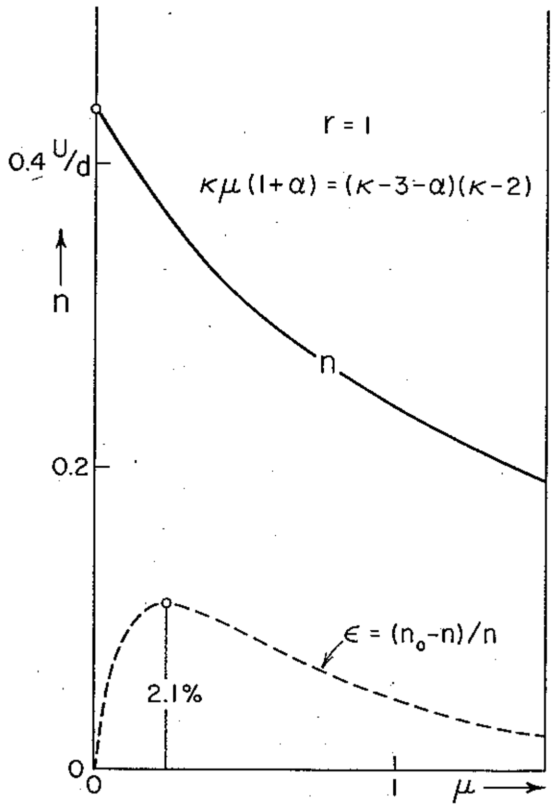

Fig. 7. Growth rate for waves on c-line.

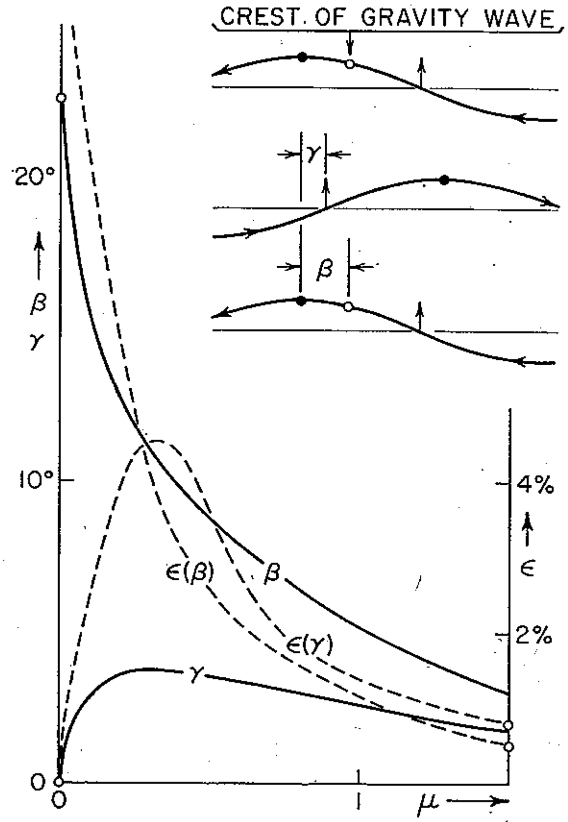

Fig. 8. Phase adjustments from c-state to growing mode.

These give

(6.6)
$$\frac{\gamma}{\beta}=\frac{1}{2}\frac{\mu U/C}{1+\mu U/C}\cdot\frac{2+3\mu U/C}{1+2\mu U/C}; \qquad \frac{\delta}{\beta\gamma}=\frac{\mu U/C}{2+3\mu U/C}.$$

The percentage change of the speed of the characteristic frame is also here an order less than the phase adjustments. The right values of γ, obtained from the theory in Appendix C, are shown in Fig. 8 by the solid line marked γ, and the percentage error $\varepsilon(\gamma)$ of the value in (6.6) is indicated by the dashed line. The errors are quite small except for small values of the static stability.

7. The (r=0)-jet. This jet has the entire static stability at the center. The theory is here practically the same as for the $(r=1)$-jet. From (4.5) the speed of the characteristic frame of the wave with a stationary c-state is given by

(7.1)
$$\kappa C_0/U=2(1+\mu U/C_0)=\kappa-1-\alpha, \qquad (C_0=U-C)$$

and its wave number is given by

(7.2)
$$(\kappa-1-\alpha)(\kappa-3-\alpha)=2\kappa\mu.$$

When the deformations have the amplitude ratio

$$A_s/A_0=\sqrt{1+\mu U/C_0}$$

they grow at the same rate

$$\dot{A}_0/A_0=\dot{A}_s/A_s=2\sqrt{\alpha}\sqrt{1+\mu U/C_0}\,(U/d).$$

After the adjustment from this *c*-state to the growing mode all wave amplitudes grow at the same rate as the gravity wave in the center

$$n=\dot{A}_s/A_s=\dot{A}_0/A_0=\dot{A}_g/A_g=kC_0\tan\beta. \tag{7.3}$$

This condition combined with the kinematic conditions at the wave crests, to the first order of the phase angles, give

$$A_s/A_0=\sqrt{1+2\mu U/C_0}\,; \qquad 2\beta=2\sqrt{\alpha}/\sqrt{1+2\mu U/C_0}. \tag{7.4}$$

and hence from (7.3) the approximate growth rate

$$n_0=2\sqrt{\alpha}\,\frac{1+\mu U/C_0}{\sqrt{1+2\mu U/C_0}}\cdot\frac{U}{d}. \tag{7.5}$$

The kinematic conditions at the deformation nodes, to the second order of the phase angles, give

$$\frac{\gamma}{\beta}=\frac{1}{2}\frac{\mu U/C_0}{1+\mu U/C_0}\cdot\frac{2+3\mu U/C_0}{1+2\mu U/C_0}\,; \qquad \frac{\delta_0}{\beta\gamma}=\frac{\mu U/C_0}{2+3\mu U/C_0} \tag{7.6}$$

where δ_0 denotes the percentage change of the speed C_0 of the characteristic frame from the *c*-state to the normal mode.

The right values of the growth rate and the phase adjustments, as obtained from the theory in Appendix C, are shown by the solid lines in Figs. 9 and 10, with the errors of the values from (7.4–6) indicated by dashed lines.

8. The stability boundaries in stratified four layer jet. The waves on these boundaries are non-tilting neutral waves, with the wave on the boundaries stationary relative to the wave in the center. The kinematic conditions at the nodes of these waves, as in (4.5), are

$$\begin{aligned} w_s&=(1+\alpha)\frac{U}{d}\left[1+\mu U\frac{r}{C}\right]A_s\pm 2\sqrt{\alpha}\frac{U}{d}\left[1+\mu U\frac{1-r}{U-C}\right]A_0=kA_sC,\\ w_0&=\pm 2\sqrt{\alpha}\frac{U}{d}\left[1+\mu U\frac{r}{C}\right]A_s+2\frac{U}{d}\left[1+\mu U\frac{1-r}{U-C}\right]A_0=kA_0(U-C). \end{aligned} \tag{8.1}$$

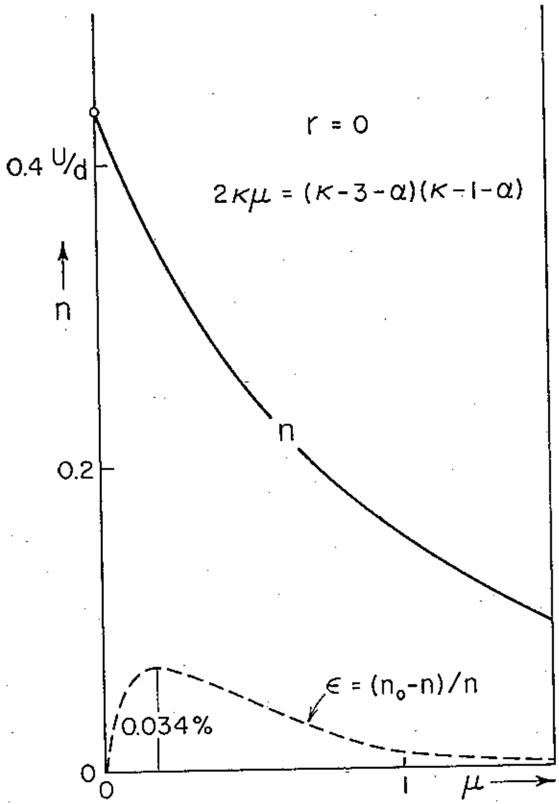

Fig. 9. Growth rate for waves on *c*-line.

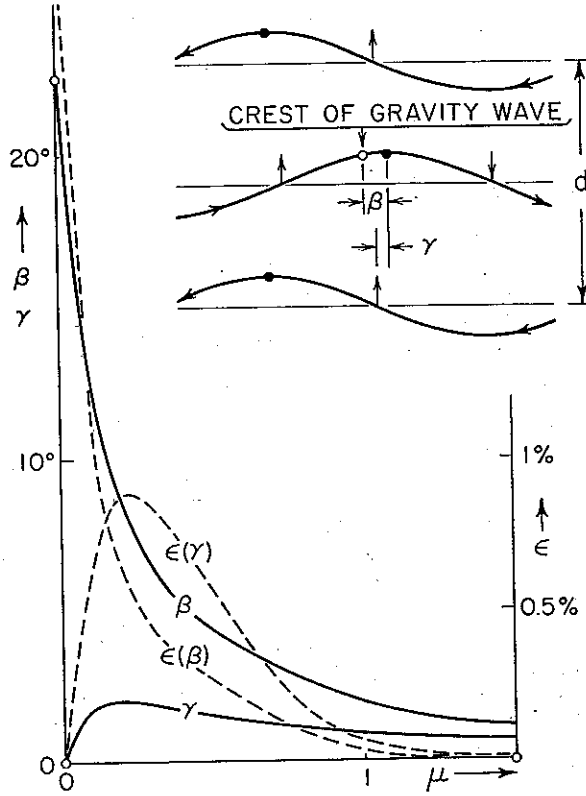

Fig. 10. Phase adjustments from *c*-state to growing mode.

To make these the conditions for the waves on the stability boundaries the amplitudes must be chosen such that the mutual interactions between the center and the boundaries give the same contribution to the intrinsic wave speeds. This condition gives the same amplitude ratio as the condition for equal growth in the stationary *c*-state in (4.8), namely

$$\left(\frac{A_0}{A_s}\right)^2=\frac{1+\mu Ur/C}{1+\mu U(1-r)/(U-C)}.$$

With this amplitude ratio the kinematic conditions in (8.1) become

$$\kappa C/U=(1+\alpha)(1+\mu Ur/C)\pm 2\sqrt{\alpha}\sqrt{1+\mu Ur/C}\cdot\sqrt{1+\mu U(1-r)/(U-C)},$$

$$\kappa(1-C/U)=2[1+\mu U(1-r)/(U-C)]\pm 2\sqrt{\alpha}\sqrt{1+\mu Ur/C}\cdot\sqrt{1+\mu U(1-r)/(U-C)}.$$

These equations determine the wave number and the speed of the characteristic frame of the waves on the spectral boundaries between the stable and the unstable waves. The positive sign for the interaction term gives a stationary *b*-state with the center deformation in opposite phase to the boundary deformation. The negative sign gives a stationary *a*-state with deformations in the same phase.

It is very easy to find the stability boundaries in a jet which has the static stability ratio $r=C/U$, where C denotes the speed of the characteristic frame. They are given by

$$\kappa C/U=(1+\alpha\pm 2\sqrt{\alpha})(1+\mu)=\kappa-(2\pm 2\sqrt{\alpha})(1+\mu).$$

The stability boundaries are therefore

$$\kappa=(3+\alpha\pm 4\sqrt{\alpha})(1+\mu), \tag{8.2}$$

and the static stability ratio is

$$r=\frac{1+\alpha\pm 2\sqrt{\alpha}}{3+\alpha\pm 4\sqrt{\alpha}}=\frac{C}{U}. \tag{8.3}$$

These boundaries are shown as dashed lines in Fig. 4. The b-line starts from $\kappa=3+\alpha+4\sqrt{\alpha}=3.67$ with the slope $d\mu/d\kappa=(1+\alpha+2/\sqrt{\alpha})/\kappa$. The a-line starts from the origin along the parabola $\kappa^2=12\mu$. Their common asymptote is the c-line of the $\left(r=\frac{1}{3}\right)$-jet, $\kappa=3(1+\mu)$. It should be noted that the static stability ratio does not have a constant value on these stability boundaries. On the b-line the value is a little larger than one-third, but only for small μ is the departure of any significance. On the a-line the static stability ratio is less than one-third and for small μ, as the a-line approaches the origin, the value decreases toward zero $(r\to\frac{1}{4}\kappa\to 0)$. Here the a-line approaches the stability boundary of the $(r=0)$-jet.

The stability boundaries of the $(r=0)$-jet are determined by the conditions

$$\kappa C_0/U=2(1+\mu U/C_0)\pm 2\sqrt{\alpha}\sqrt{1+\mu U/C_0} \qquad (C_0=U-C)$$

$$\kappa(1-C_0/U)=1+\alpha\pm 2\sqrt{\alpha}\sqrt{1+\mu U/C_0}.$$

The difference of these give one quadratic equation for C_0/U, whose positive root is

$$4\kappa C_0/U=(\kappa+1-\alpha)+\sqrt{(\kappa+1-\alpha)^2+16\kappa\mu}. \qquad (r=0)$$

The sum of the two conditions gives a quadratic equation for $\sqrt{1+\mu U/C_0}$. The positive root substituted in one of the conditions gives a second expression for C_0/U, namely

$$\kappa C_0/U=\kappa-1+\alpha\pm\sqrt{2\alpha(\mu-1+\alpha)}. \qquad (r=0)$$

Elimination of C_0 gives the stability boundaries in the $(r=0)$-jet

$$2\kappa\mu=(\kappa-1+\alpha)(\kappa-3-\alpha)\pm\sqrt{2\alpha(\kappa-1+\alpha)[(\kappa+1-\alpha)^2+16\kappa\mu]}. \tag{8.4}$$

Their common asymptote is the hyperbolic c-line for the $(r=0)$-jet, $2\kappa\mu=(\kappa-1)(\kappa-3)$.

The b-line starts from $\kappa=3+\alpha+4\sqrt{\alpha}$. The a-line starts from the origin along the parabola $\kappa^2=12\mu$.

The conditions for the stability boundaries in the $(r=1)$-jet are

$$\kappa C/U=(1+\alpha)(1+\mu U/C)\pm 2\sqrt{\alpha}\sqrt{1+\mu U/C},$$

$$\kappa(1-C/U)=2\pm 2\sqrt{\alpha}\sqrt{1+\mu U/C}.$$

The same procedure as above gives two explicit expressions for C, namely

$$4\kappa C/U=\kappa-1+\alpha+\sqrt{(\kappa-1+\alpha)^2+8(1+\alpha)\kappa\mu},$$

$$(1+\alpha)\kappa C/U=4\alpha+(1+\alpha)(\kappa-2)\pm 2\sqrt{\alpha}\sqrt{4\alpha(1+\alpha)(\kappa-2)}.$$

From these, by elimination of C, come the equations for the stability boundaries in the $(r=1)$-jet,

$$4[4\alpha+(1+\alpha)(\kappa-2)\pm 2\sqrt{\alpha}\sqrt{\alpha+(1+\alpha)(\kappa-2)}]=$$

$$=(1+\alpha)[(\kappa-1+\alpha)+\sqrt{(\kappa-1+\alpha)^2+8(1+\alpha)\kappa\mu}].$$

The common asymptote of the stability boundaries is the hyperbolic c-line for the $(r=1)$-jet, $\kappa\mu=(\kappa-2)(\kappa-3)$. The b-line starts from $\kappa=3+\alpha+4\sqrt{\alpha}$. The a-line starts from the origin along the parabola $\kappa^2=6\mu$.

The stability boundaries may be obtained from the theory in Appendix C, by using the condition that the frequency equation here has a real double root, $m=C/U$, which makes the discriminant of the equation zero. However, the derivation is more laborious and does not reveal the simple property of the amplitude ratio on the stability boundaries.

The stability boundaries for the $(r=1)$-jet and for the $(r=1/3)$-jet are shown in Fig. 11. The broken lines between the boundaries are the corresponding c-lines. The growth rates of the normal modes on the c-lines are shown in the left part of the Figure. The jet is most unstable when the entire static stability is located at the boundaries $(r=1)$. The instability is weakest when two-thirds of the static stability is located in the center of the jet. For intermediate values of the static stability ratio the growth rate of the normal mode on the c-line may be obtained with an accuracy of a few per cent by linear interpolation between these extreme values.

APPENDIX A

Normal modes of the homogeneous shear layer

In an arbitrary state of a symmetric wave the boundaries of the shear layer have the deformations:

Upper deformation: $z_1=A\cos(kx-\sigma)$,
Lower deformation: $z_2=A\cos(kx+\sigma)$.

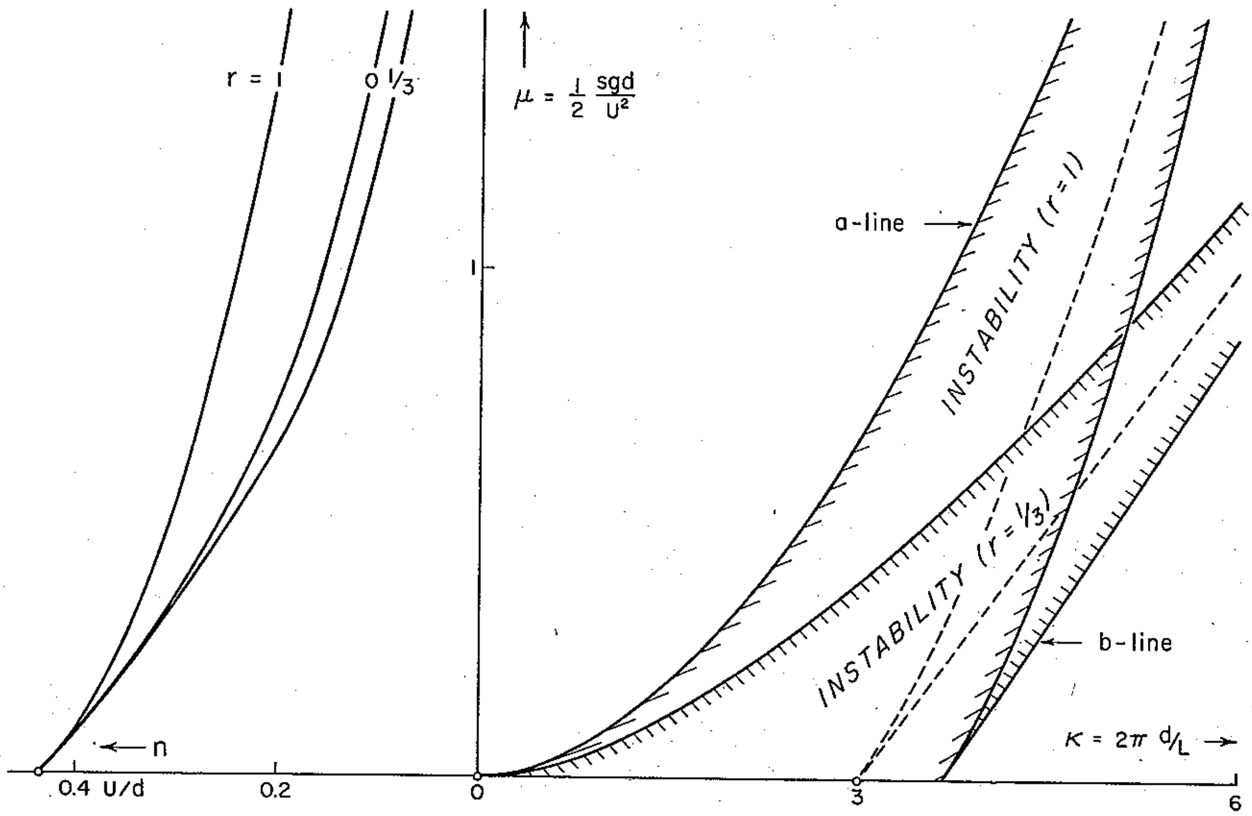

Fig. 11. Stability boundaries in the stratified jet.

With the notations $d=$ depth of shear layer and $2U=$ relative speed of outer layers, the kinematic and dynamic conditions at the upper boundary are

$$w_1=\left(\frac{\partial}{\partial t}+U\frac{\partial}{\partial x}\right)z_1=\frac{U}{kd}\frac{\partial}{\partial x}(z_1-e^{-kd}z_2).$$

Substitute here the deformations and introduce the abbreviations

(A.1) $$kd=\kappa,\ e^{-kd}=\alpha,\ n_a=(1-\alpha)/\kappa;\qquad n_b=(1+\alpha)/\kappa.$$

The resulting evolution equations for the symmetric wave are

$$\frac{\frac{d}{dt}(A\sin\sigma)}{A\cos\sigma}=(1-n_a)kU=\frac{\dot{A}}{A}\tan\sigma+\dot{\sigma}=n\tan\sigma_s=\dot{\sigma}_a(\sigma=0).$$

$$\frac{\frac{d}{dt}(A\cos\sigma)}{A\sin\sigma}=(n_b-1)kU=\frac{\dot{A}}{A}\cot\sigma-\dot{\sigma}=n\cot\sigma_s=-\dot{\sigma}_b(\sigma=90^0).$$

The phase speed $\dot{\sigma}$ in an arbitrary state is related to the speeds in the non-tilting states by the formula

$$(A.2) \qquad \dot{\sigma}=\dot{\sigma}_a\cos^2\sigma+\dot{\sigma}_b\sin^2\sigma.$$

The growth rate n and phase σ_s in the stationary growing state of the wave (the normal mode) are related to the phase speeds in the non-tilting states by the formulas

$$(A.3) \qquad n^2=-\dot{\sigma}_a\dot{\sigma}_b=-(1-n_a)(1-n_b)(kU)^2=[\alpha^2-(1-\kappa)^2](U/d)^2$$

$$(A.4) \qquad \tan^2\sigma_s=-\dot{\sigma}_a\,\dot{\sigma}_b; \qquad \cos 2\sigma_s=(1-\kappa)/\alpha.$$

The fastest growing mode $(dn/d\alpha=0)$ has the wave number $\kappa=1-\alpha^2=0.8$, the growth rate $n=0.4U/d$, and the tilt $2\sigma_s=63°\,30'$.

APPENDIX B

Normal modes of the homogeneous four-layer jet

We use the notations $d=$ depth of the jet, $U=$ speed of fluid in center of jet relative to unbounded outer fluid. In an arbitrary state of the wave the center and the boundaries of the jet have the deformations

Center: $z_0=A\cos(kx-\sigma)$.
Boundaries: $z_s=A\cos(kx+\sigma)$.

They have the same amplitudes to give equal interactions between the center and the boundaries.

With the abbreviations in (2.1): $\kappa=kd$, $\alpha=e^{-\kappa}$, the speed of the characteristic frame of the wave relative to the outer fluid (see 2.3 and 2.6) is

$$C=\tfrac{1}{2}\left(1-\frac{1-\alpha}{\kappa}\right)U.$$

The kinematic and dynamic conditions at the boundaries and the center are

Boundaries: $$w_s=\left[\frac{\partial}{\partial t}-C\frac{\partial}{\partial x}\right]z_s=\frac{U}{kd}\frac{\partial}{\partial x}[-(1+\alpha)z_s+2\sqrt{\alpha}\,z_0].$$

Center: $$w_0=\left[\frac{\partial}{\partial t}+(U-C)\frac{\partial}{\partial x}\right]z_0=\frac{U}{kd}\frac{\partial}{\partial x}[2z_0-2\sqrt{\alpha}\,z_s].$$

With the value of C substituted the local changes in the characteristic frame are

$$\frac{\kappa}{U}\frac{\partial z_s}{\partial t}=\tfrac{1}{2}(\kappa-3-\alpha)\frac{\partial z_s}{\partial x}+2\sqrt{\alpha}\frac{\partial z_0}{\partial x},$$

$$-\frac{\kappa}{U}\frac{\partial z_0}{\partial t}=\tfrac{1}{2}(\kappa-3-\alpha)\frac{\partial z_0}{\partial x}+2\sqrt{\alpha}\frac{\partial z_s}{\partial x}.$$

Substitute here the deformations, and introduce the abbreviations

(B.1) $$n_a=(3+\alpha-4\sqrt{\alpha})/\kappa; \qquad n_b=(3+\alpha+4\sqrt{\alpha})/\kappa.$$

Both equations give the same evolution equation for the symmetric wave, namely

(B.2)
$$\frac{\frac{d}{dt}(A\sin\sigma)}{A\cos\sigma}=\tfrac{1}{2}(1-n_a)kU=\frac{\dot{A}}{A}\tan\sigma+\dot{\sigma}=n\tan\sigma_s=\dot{\sigma}_a(\sigma=0)$$

$$\frac{\frac{d}{dt}(A\cos\sigma)}{A\sin\sigma}=\tfrac{1}{2}(n_b-1)kU=\frac{\dot{A}}{A}\cot\sigma-\dot{\sigma}=n\cot\sigma_s=-\dot{\sigma}_b(\sigma=90^0).$$

They are formally identical to the evolution equations for the symmetric wave in the shear layer (Appendix A), only n_a and n_b are different functions of the wave length. The growth rate n and the phase σ_s in the stationary growing state of the wave (the normal mode) are also here related to the phase speeds in the non-tilting states by the formulas

(B.3) $$n^2=-\dot{\sigma}_a\dot{\sigma}_b=-\tfrac{1}{4}(1-n_a)(1-n_b)(kU)^2=[4\alpha-\tfrac{1}{4}(3+\alpha-\kappa)^2](U/d)^2$$

(B.4) $$\tan^2\sigma_s=-\dot{\sigma}_a/\dot{\sigma}_b, \qquad \cos 2\sigma_s=\tfrac{1}{4}(3+\alpha-\kappa)/\sqrt{\alpha}.$$

The fastest growing wave $(dn/d\alpha=0)$ has the wave number $\kappa=3+\alpha-8\alpha/(1+\alpha)=2.45$, the growth rate $n=2\sqrt{\alpha}(1-\alpha)/(1+\alpha)(U/d)=0.495U/d$, and the tilt $\cos 2\sigma_s=2\sqrt{\alpha}/(1+\alpha)$, or $2\sigma_s=57°$.

Besides this unstable pair of modes the jet has a third stable mode with a node at the center level and the boundaries deformed in opposite phase.

APPENDIX C

Normal modes of the stratified jet

We use the notations in section 4, and further z_i= deformation of interface $(i=1,0,2)$. $(u^+-u^-)_i=(2U/d)g_i$= gravity wave in interface. These are related through the dynamic condition

$$\frac{Dg_i}{Dt}=\frac{\sigma g d}{U}\frac{\partial z_i}{\partial x}. \tag{C.1}$$

In a frame fixed to the outer fluid the wave elements of the normal mode are

$$z_i, g_i \sim \exp ik(x-mUt); \qquad (m=C/U+in/Uk). \tag{C.2}$$

From (C.1) the gravity waves in the boundaries are

$$g_{1,2}=-(r/m)\mu z_{1,2}; \qquad (r=2\sigma_s/s).$$

and the gravity wave in the center of the jet is

$$g_0=2(1-r)/(1-m)\mu z_0.$$

The kinematic conditions at the boundaries and center of the jet are

$$\frac{Dz_1}{Dt}=-mU\frac{\partial z_1}{\partial x}=\frac{U}{kd}\frac{\partial}{\partial x}[(g_1-z_1)+\alpha(g_2-z_2)+\sqrt{\alpha}(g_0+2z_0)],$$

$$\frac{Dz_2}{Dt}=-mU\frac{\partial z_2}{\partial x}=\frac{U}{kd}\frac{\partial}{\partial x}[(g_2-z_2)+\alpha(g_1-z_1)+\sqrt{\alpha}(g_0+2z_0)],$$

$$\frac{Dz_0}{Dt}=(1-m)U\frac{\partial z_0}{\partial x}=\frac{U}{kd}\frac{\partial}{\partial x}[(g_0+2z_0)+\sqrt{\alpha}(g_1-z_1+g_2-z_2)].$$

The frequency equation is obtained by putting the determinant of this system equal to zero. It may be written in the form

$$\left[\kappa m-(1-\alpha)\left(1+\frac{r}{m}\mu\right)\right]\left[\left\{\kappa m-(1+\alpha)\left(1+\frac{r}{m}\mu\right)\right\}\left\{\kappa m-\kappa+2\left(1+\frac{1-r}{1-m}\mu\right)\right\}+\right.$$
$$\left.+4\alpha\left(1+\frac{r}{m}\mu\right)\left(1+\frac{1-r}{1-m}\mu\right)\right]=0.$$

It is the product of a quadratic factor which determines a pair of stable modes with a nodal plane in the center and the boundaries having equal deformations in opposite phase, and a quartic factor which determines two pairs of modes with the boundaries having equal deformations in the same phase.

For $\mu=0$ the quartic reduces to a quadratic whose roots, $m=C/U\pm in/Uk$, give the values of C and n in Appendix B.

For waves on the c-line in section 5 with the wave numbers $\kappa=(3+\alpha)(1+\mu)$ in a jet with the static stability ratio $r=(1+\alpha)/(3+\alpha)$, the frequency equation has a pair of complex roots with the real part $C/U=r$, as predicted in (5.1). The imaginary part gives the growth n/kU which is shown in Fig. 5.

In the $(r=1)$-jet the quartic factor of the frequency equation becomes cubic. For waves on the c-line of this jet, $(1+\alpha)\kappa\mu=(\kappa-2)(\kappa-3-\alpha)$, it has a conjugate complex pair of roots whose real part has the value $C/U=1-2/\kappa$, as predicted in (6.1). The imaginary part is the growth rate which is shown in Fig. 7.

Similarly in the $(r=0)$-jet the quartic factor of the frequency equation becomes cubic. For waves on the c-line, $2\kappa\mu=(\kappa-1-\alpha)(\kappa-3-\alpha)$, it has a conjugate complex pair of roots whose real part has the value $C/U=(1+\alpha)/\kappa$ in (7.1). The imaginary part is the growth rate shown in Fig. 9.

LIST OF SYMBOLS

d	Depth of the shear layer and jet.
U	In shear layer, one-half of relative speed between outer layers.
U	In jet, relative speed between center fluid and outer fluid.
k	Wave number.
κ	kd
α	e^{-kd}
n	Growth rate of normal mode.
ρ_2	Density of lower layer.
ρ_1	Density of upper layer.
s	$(\rho_2-\rho_1)/(\rho_2+\rho_1)$
μ	$\frac{1}{2}sgd/U^2$, Richardson number.
$\Delta\rho_s$	Density jump across boundaries of jet.
$\Delta\rho_0$	Density jump across center of jet.
σ_s	$\Delta\rho_s/(\rho_2+\rho_1)$
σ_0	$\Delta\rho_0/(\rho_2+\rho_1)$
r	$2\sigma_s/s$, static stability ratio in jet.
A,A_s	Amplitude of boundary deformations.
A,A_0	Amplitude of center deformation.
A_g	Amplitude of gravity wave.
C,C_s	Intrinsic phase speed of boundary wave.
C_0	Intrinsic phase speed of center wave.
C	Speed of characteristic frame relative to outer fluid.

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